Lectures on Superstring Amplitudes

Part 2: Superstrings

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Superstring Perturbation Theory

- **Theory of fluctuating random surfaces** (closed strings shown)
  
  - governed by topological expansion in the genus $h$ weighed by $g_s^{2h-2}$
  
  \[
  g_s^{-2} + g_s^0 + g_s^2 + \cdots
  \]

- **Bosonic string**
  
  - unstable with closed string tachyon
  
  - Nature has fermions!

- **Superstrings generalize bosonic string**
  
  - they have fermions
  
  - no tachyon
  
  - supersymmetry
Approaches to Superstring Perturbation Theory

• Goal is to obtain superstring amplitudes at all genera
  – Ramond-Neveu-Schwarz formulation of fermionic strings; w/ Gliozzi-Scherk-Olive projection to supersymmetric spectrum;
  – Green-Schwarz space-time supersymmetric formulation;
  – Mandelstam light-cone formulation;
  – String field theory;
  – Topological string theory;
  – Berkovits pure spinor formulation.

• Different perturbative superstring theories (in 10 dimensions)
  – Type I open & closed, orientable & non-orientable, D-branes
  – Type IIA,B closed orientable, D-branes
  – Heterotic closed orientable $E_8 \times E_8, \text{Spin}(32/\mathbb{Z}_2)$

• Here: RNS formulation, closed orientable superstrings, dimension 10
Genus-zero four-graviton superstring amplitude

• Kinematics of the four-graviton amplitude
  – momenta of gravitons $k_i^\mu$ are conserved $\sum_i k_i^\mu = 0$
  – choose basis of factorized polarization tensors $\varepsilon_i^{\mu\nu} = \varepsilon_i^\mu \tilde{\varepsilon}_i^\nu$
  – masslessness $k_i^2 = 0$ and transversality $k_i^\mu \varepsilon_i^\mu = k_i^\mu \tilde{\varepsilon}_i^\mu = 0$ for $i = 1, 2, 3, 4$
  – kinematic invariants $s = s_{12} = s_{34}$, $t = s_{14} = s_{23}$, $u = s_{13} = s_{24}$

  $$s_{ij} = -\alpha'(k_i + k_j)^2/4$$

• Tree-level four-graviton amplitude is given by

$$A^{(0)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = \frac{1}{g_s^2} \times \mathcal{K}\tilde{\mathcal{K}} \times \frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)}$$

  – Kinematical factor $\mathcal{K}$ given in terms of $f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - k_i^\nu \varepsilon_i^\mu$ by

  $$\mathcal{K} = (f_1f_2)(f_3f_4) + (f_1f_3)(f_2f_4) + (f_1f_4)(f_2f_3) - 4(f_1f_2f_3f_4) - 4(f_1f_2f_4f_3) - 4(f_1f_3f_2f_4)$$

  – for $\tilde{\mathcal{K}}$ replace $\varepsilon_i$ by $\tilde{\varepsilon}_i$

  – Equivalently, $\mathcal{K} \times \tilde{\mathcal{K}} = R^4$ with $R$ the linearized Weyl tensor

  – String duality: symmetric in $s, t, u$

  – Poles in each channel, at $s, t, u = 0, 1, 2, \cdots$
Genus-one four-graviton superstring amplitude

- **Type II four-graviton amplitude to one-loop order** (Green, Schwarz 1982)

\[ A^{(1)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = \mathcal{R}^4 \int_{\mathcal{M}_1} \frac{d^2 \tau}{(\text{Im} \, \tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau) \]

- Partial amplitude \( \mathcal{B}^{(1)} \) is a modular function in \( \tau \in \mathcal{M}_1 = \mathcal{H}_1/SL(2, \mathbb{Z}) \)

\[ \mathcal{B}^{(1)}(s_{ij}|\tau) = \int_{\Sigma^4} \prod_{i=1}^{4} \frac{d^2 z_i}{\text{Im} \, \tau} \exp \left( \sum_{i<j} s_{ij} G(z_i - z_j|\tau) \right) \]

- \( G(z|\tau) \) is the scalar Green function on the torus \( \Sigma \) of modulus \( \tau \).
- Analogous formulas for Heterotic strings and more external states.

- **Singularity structure**
  - For fixed \( \tau \) integrations over \( \Sigma \) produce poles in \( \mathcal{B}^{(1)} \) at positive integers \( s_{ij} \).
  - The integral over \( \tau \) converges absolutely only for \( \text{Re}(s_{ij}) = 0 \).
  - Analytic continuation to \( s_{ij} \in \mathbb{C} \) via decomposition of \( \mathcal{M}_1 \).
  - Branch cuts in \( s_{ij} \) starting at integers \( \geq 0 \) are produced by \( \tau \to i\infty \) region.
Loop momenta

- Loop momenta may be exposed
  - Choose a canonical basis of homology cycles $\mathcal{A}, \mathcal{B}$.
  - Choose loop momentum $p$ flowing through the cycle $\mathcal{A}$,
    \[ \int_{M_1} \frac{d^2\tau}{(\text{Im}\,\tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau) = \int_{\mathbb{R}^{10}} d^{10}p \int_{M_1} \int_{\Sigma^4} |\mathcal{F}(z_i, k_i, p|\tau)|^2 \]

- Chiral amplitude $\mathcal{F}$ is locally holomorphic in $\tau$ and $z_i$
  \[ \mathcal{F}(z_i, k_i, p|\tau) = e^{i\pi p^2 + 2\pi ip \sum_i k_i z_i} \prod_{i<j} \vartheta_1(z_i - z_j|\tau)^{-s_{ij}} d\tau \prod_{i=1}^{4} d\bar{z}_i \]
  at the cost of non-trivial monodromy
  \[ \mathcal{F}(z_i + \delta_{i,\ell} \mathcal{A}, k_i, p|\tau) = e^{2\pi ik_{\ell \cdot p}} \mathcal{F}(z_i, k_i, p|\tau) \]
  \[ \mathcal{F}(z_i + \delta_{i,\ell} \mathcal{B}, k_i, p|\tau) = \mathcal{F}(z_i, k_i, p + k_{\ell}|\tau) \]

- Modular invariance of $\mathcal{A}^{(1)}$ guarantees independence of choices.
- Hermitian pairing of $\mathcal{F}$ and $\bar{\mathcal{F}}$ is familiar from 2-d CFT where loop momentum $p$ labels conformal blocks of 10 copies of $c = 1$. 

**UV-finiteness**

- **Thanks to modular invariance, all string amplitudes are UV-finite**
  - shown for the closed bosonic string at genus one (Shapiro 1972)
  - holds for all modular invariant superstrings to all loops (i.e. all genera)

- **For genus-one: All chiral amplitudes have a universal factor**

  $$\mathcal{F}(z_i, \epsilon_i, k_i, p_I | \tau) = e^{ip\mu \tau \rho^\mu} \times \cdots$$

  - Modular invariance allows one to choose a fundamental domain where $\text{Im}(\tau)$ bounded from below

  $$\mathcal{H}_1/SL(2, \mathbb{Z}) = \{ \tau \in \mathbb{C}, \text{Im}(\tau) > 0, |\tau| \geq 1, |\text{Re}(\tau)| \leq \frac{1}{2} \}$$

  - Analogous, more complicated, choices to higher genus

  $\Rightarrow$ **Uniform Gaussian suppression at large loop momenta**

  $\Rightarrow$ **UV finiteness**
RNS formulation of superstrings

- $M = \mathbb{R}^{10}$ flat Minkowski space-time with Lorentz group $SO(1, 9)$
  - $x^\mu$ scalars on worldsheet $\Sigma$, map $\Sigma$ into $M$
  - $\psi^\mu$ spinors on $\Sigma$ but Lorentz vector under $SO(1, 9)$
    - Worldsheet supersymmetry $\implies \Sigma$ is a super Riemann surface
    - Two sectors: NS bosons $SO(1, 9)$-tensors
      - R fermions $SO(1, 9)$-spinors

- With Minkowski signature $\Sigma$
  - $\psi^\mu$ and $\tilde{\psi}^\mu$ are independent Majorana-Weyl spinors of opposite chirality

- With Euclidean signature $\Sigma$
  - $\psi^\mu$ and $\tilde{\psi}^\mu$ must be independent complex Weyl spinors
  - Globally, on a compact Riemann surface of genus $h$,
    - All $\psi^\mu$ are sections of a the same spin bundle $S$ (and $\tilde{\psi}^\mu$ of $\tilde{S}$)
    - $2^{2h}$ distinct spin structures for $S$ (and $2^{2h}$ independently for $\tilde{S}$)

- GSO projection requires independent summation over spin structures
Quantization of worldsheet spinor fields

- **Illustrate**
  - Ramond and Neveu-Schwarz sectors
  - independence of chiralities

- **Dirac action and equation for flat** $M = \mathbb{R}^{10}$ **with metric** $\eta$
  - All components of $\psi^\mu_+$ are sections of the same spin bundle $S$
  - Complex structure $J$ with local complex coordinates $(z, \bar{z})$
  - Dirac action,
    \[ I_\psi[\psi, J] = \frac{1}{2\pi} \int_\Sigma d\bar{z}dz \psi^\mu_+ \partial_\bar{z} \psi^\nu_+ \eta_{\mu\nu} \]
  - Dirac equation $\partial_\bar{z} \psi^\mu_+ = 0$ has locally holomorphic solutions,
  - but products of operators produce singularities
    \[ \psi^\mu_+(z) \psi^\nu_+(w) = \frac{\eta^\mu\nu}{z - w} + \text{regular} \]
  - each component $\psi^\mu$ generates a CFT with central charge $c = \frac{1}{2}$. 


Quantization of worldsheet spinor fields (cont’d)

- **Quantization on flat cylinder or conformal equivalent flat annulus**
  - cylinder \( w = \tau + i\sigma \) with identification \( \sigma \approx \sigma + 2\pi \)
  - annulus centered at \( z = 0 \), conformally mapped by \( z = e^w \)
  - one-forms related by \( dz = e^w \, dw \), spinors by \( (dz)^{1/2} = e^{w/2} (dw)^{1/2} \)
  - fields related by conformal transformation \( \psi_{\text{cyl}}(z) = e^{w/2} \psi_{\text{ann}}(w) \)

- **Two possible spin structures**
  - NS \( \psi^\mu_{\text{cyl}}(\tau, \sigma + 2\pi) = - \psi^\mu_{\text{cyl}}(\tau, \sigma) \) or \( \psi^\mu_{\text{ann}}(e^{2\pi i} \, z) = + \psi^\mu_{\text{ann}}(z) \)
  - R \( \psi^\mu_{\text{cyl}}(\tau, \sigma + 2\pi) = + \psi^\mu_{\text{cyl}}(\tau, \sigma) \) or \( \psi^\mu_{\text{ann}}(e^{2\pi i} \, z) = - \psi^\mu_{\text{ann}}(z) \)

- **Free field quantization in annulus representation**
  - NS \( \psi^\mu(z) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} b^\mu_r \, z^{-\frac{1}{2} - r} \quad \{ b^\mu_r, b^\nu_s \} = \eta^{\mu\nu} \delta_{r+s,0} \)
  - R \( \psi^\mu(z) = \sum_{n \in \mathbb{Z}} d^\mu_n \, z^{-\frac{1}{2} - n} \quad \{ d^\mu_m, d^\nu_n \} = \eta^{\mu\nu} \delta_{m+n,0} \)
Quantization of worldsheet spinor fields (cont’d)

• Lorentz generators of $SO(1,9)$:

\[
[J^{\mu\nu}, \psi^\kappa(z)] = \eta^{\nu\kappa} \psi^\mu(z) - \eta^{\mu\kappa} \psi^\nu(z)
\]

\[
J_{NS}^{\mu\nu} = \sum_{r \in \mathbb{N} - \frac{1}{2}} \left( b_{-r}^\mu b_r^\nu - b_{-r}^\nu b_r^\mu \right)
\]

\[
J_R^{\mu\nu} = \frac{1}{2} [d_0^{\mu}, d_0^{\nu}] + \sum_{n \in \mathbb{N}} (d_{-n}^{\mu} d_n^{\nu} - d_{-n}^{\nu} d_n^{\mu})
\]

• Fock space construction produces two sectors

★ NS ground state defined by $b_r^\mu |0; NS\rangle = 0$ for all $r > 0$
  – $|0; NS\rangle$ is unique and in trivial representation of $SO(1,9)$
  – Fock space = linear combinations of $b_{-r_1}^{\mu_1} \cdots b_{-r_p}^{\mu_p} |0; NS\rangle$, $r_i > 0$
  – All states in tensor reps of $SO(1,9)$ are space-time bosons.

★ R ground state defined by $d_n^{\mu} |0, \alpha; R\rangle = 0$ for all $n > 0$
  – $|0, \alpha; R\rangle$ is degenerate and in spinor rep. of $SO(1,9)$, states labelled by $\alpha$
  – Fock space = linear combinations of $d_{-n_1}^{\mu_1} \cdots d_{-n_p}^{\mu_p} |0, \alpha; R\rangle$, $n_i > 0$
  – All states in spinor reps of $SO(1,9)$ are space-time fermions.
Summation over spin structures

- Theory with bosons and fermions requires both NS and R sectors
  - to include both, one must sum over two spin structures of the annulus

- Type II spin structures of $\psi^\mu_{\pm}$ are independent of one another
  - space-time fermions are in the $R \otimes NS$ and $NS \otimes R$ sectors
    which could never arise if spin structures for opposite chiralities coincided

- On the torus, viewed as cylinder + identification
  - spin structures along cycle of cylinder produce R and NS sectors
  - sum over spin structures along conjugate cycle produces GSO-projection
    * reduces to half the states in both R and NS sectors
    * R-sector: space-time spinor of definite chirality
    * NS-sector: eliminates the tachyon
  $\Rightarrow$ sum over all spin structures
Summation over spin structures (cont’d)

- Fix a canonical homology basis of cycles $\mathcal{A}_I, \mathcal{B}_I$ of $H_1(\Sigma, \mathbb{Z})$ $I = 1, \ldots, h$
  - with canonical intersection pairing
    $\#(\mathcal{A}_I, \mathcal{A}_J) = \#(\mathcal{B}_I, \mathcal{B}_J) = 0$ and $\#(\mathcal{A}_I, \mathcal{A}_J) = \delta_{IJ}$

- Transformations which maps one canonical basis into another
  - linear with integer coefficients
  - preserve the intersection matrix: $Sp(2h, \mathbb{Z})$

- On Riemann surface of higher genus $h$ sum over all spin structures
  - along $\mathcal{A}$-cycles produces R and NS sectors
  - along $\mathcal{B}$-cycles produces GSO-projection
  - mapped into one another by $Sp(2h, \mathbb{Z}_2)$
Super Riemann surfaces

• **Ordinary Riemann surface** (locally $\mathbb{C}$ with coordinate $z$)
  - complex manifold: holomorphic transition functions $z \to z'(z)$;
  - complex structure = conformal structure $J$
  - Moduli space $\mathcal{M}_h = \{J\}/\text{Diff}(\Sigma)$ of genus $h$ compact Riemann surfaces

• **Complex super manifold** (locally $\mathbb{C}^{1|1}$ with coordinates $z|\theta$)
  - holó transition functions $z|\theta \to z'(z, \theta)|\theta'(z, \theta)$ generate $\mathcal{N} = 2$ super conformal

• **Super Riemann surface** (locally $\mathbb{C}^{1|1}$ with coordinates $z|\theta$)
  - holó transition functions $z|\theta \to z'|\theta'$ rescale $D\theta = \partial\theta + \theta\partial z$
  - Transition functions define $\mathcal{N} = 1$ superconformal structure $\mathcal{J}$
  - Globally: $T\Sigma$ has a completely non-integrable subbundle of rank $0|1$

• **Moduli space of compact super Riemann surfaces**: $\mathcal{M}_h = \{\mathcal{J}\}/\text{Diff}(\Sigma)$
  = equivalence classes of superconformal structures $\mathcal{J}$

  $$\dim_{\mathbb{C}} \mathcal{M}_h = \begin{cases} 
  0|0 & h = 0 \\
  1|0 \text{ or } 1|1 & h = 1 \text{ even or odd spin structure} \\
  3h - 3|2h - 2 & h \geq 2
  \end{cases}$$

  - odd modulus at $h = 1$ odd spin structure is a book keeping device;
  - odd moduli really first appear at genus 2, as curved super spaces.
Superstring worldsheets and moduli spaces

• Heterotic
  – Left : RS \( \Sigma_L \), moduli space \( \mathcal{M}_L \) coord resp. \( \tilde{z} \) and \( \tilde{m}^i \)
  – Right : SRS \( \Sigma_R \), moduli space \( \mathcal{M}_R \) coord resp. \( (z, \theta) \) and \( (m^i, \zeta^\alpha) \)
  – Worldsheet is a cycle \( \Sigma \subset \Sigma_L \times \Sigma_R \) of dim \( 1|1 \)
    subject to \( \Sigma_{\text{red}} = \text{diag}(\Sigma_{L \text{red}} \times \Sigma_{R \text{red}}) : \tilde{z}^* = z + \text{nilpotent} \)
  – Moduli space is a cycle \( \Gamma \subset \mathcal{M}_L \times \mathcal{M}_R \) of dim \( 3h - 3|2h - 2 \) for \( h \geq 2 \)
    subject to \( \Gamma_{\text{red}} = \text{diag}(\mathcal{M}_{L \text{red}} \times \mathcal{M}_{R \text{red}}) : (\tilde{m}^i)^* = m^i + \text{nilpotent} \)
    (reduced space obtained by setting all nilpotent variables to zero)

• Type II
  – Left : SRS \( \Sigma_L \), moduli space \( \mathcal{M}_L \) coord resp. \( (\tilde{z}, \tilde{\theta}) \) and \( (\tilde{m}^i, \tilde{\zeta}^\alpha) \)
  – Right : SRS \( \Sigma_R \), moduli space \( \mathcal{M}_R \) coord resp. \( (z, \theta) \) and \( (m^i, \zeta^\alpha) \)
  – Worldsheet is a cycle \( \Sigma \subset \Sigma_L \times \Sigma_R \) of dim \( 1|2 \)
  – Moduli space is cycle \( \Gamma \subset \mathcal{M}_L \times \mathcal{M}_R \) of dim \( 3h - 3|4h - 4 \) for \( h \geq 2 \)
    subject to \( \tilde{z}^* = z + \text{nilpotent} \) and \( (\tilde{m}^i)^* = m^i + \text{nilpotent} \)

• Super-Stokes theorem ensures independence of the choice of cycles
  – in amplitudes with BRST invariant vertex operators
  – consistent definition of superstring amplitudes to all genera (Witten 2012)
Worldsheet action for Type II superstrings

- **Worldsheet is** $\Sigma \subset \Sigma_L \times \Sigma_R$
  - $\Sigma_L$ has superconformal structure $\tilde{J}$ with local coordinates $\tilde{z}|\tilde{\theta}$
  - $\Sigma_R$ has superconformal structure $J$ with local coordinates $z|\theta$

- **Superconformal invariant matter action**
  - worldsheet matter field
    $X^\mu(\tilde{z}, z|\tilde{\theta}, \theta) = x^\mu(\tilde{z}, z) + \theta \psi^\mu(\tilde{z}, z) + \tilde{\theta} \tilde{\psi}^\mu(\tilde{z}, z) + \tilde{\theta} \theta F^\mu(\tilde{z}, z)$
  - Worldsheet action in local coordinates ($D_{\theta} = \partial_{\theta} + \theta \partial_z$)
    $I_m[X^\mu, \tilde{J}, J] = \int_{\Sigma} [d\tilde{z}dz|d\tilde{\theta}d\theta] \tilde{D}_{\tilde{\theta}} X^\mu D_{\theta} X_\mu$
  - Superconformal algebra on fields generated by
    $S_{z\theta} = S_{z\theta} + \theta T_{zz}$
    $S_{z\tilde{\theta}} = \frac{1}{2} \psi^\mu \partial_z x_\mu$
    $T_{zz} = -\frac{1}{2} \partial_z x^\mu \partial_z x_\mu + \frac{1}{2} \psi^\mu \partial_z \psi_\mu$
    $\tilde{S}_{\tilde{z}\tilde{\theta}} = \tilde{S}_{\tilde{z}\tilde{\theta}} + \tilde{\theta} \tilde{T}_{\tilde{z}\tilde{z}}$
    $\tilde{S}_{\tilde{z}\tilde{\theta}} = \frac{1}{2} \tilde{\psi}^\mu \partial_{\tilde{z}} x_\mu$
    $\tilde{T}_{\tilde{z}\tilde{z}} = -\frac{1}{2} \partial_{\tilde{z}} x^\mu \partial_{\tilde{z}} x_\mu + \frac{1}{2} \tilde{\psi}^\mu \partial_{\tilde{z}} \tilde{\psi}_\mu$
Deformations of superconformal structures

• Under deformation of $\tilde{J}$ for $\Sigma_L$ and $J$ for $\Sigma_R$

$$\delta I = \int_{\Sigma} [d\tilde{z}dz|d\tilde{\theta}d\theta] \left( H_{\tilde{\theta}}\tilde{z}^{\tilde{z}} S_{z\theta} + \tilde{H}_{\theta}^{\tilde{z}} \tilde{S}_{\tilde{z}\tilde{\theta}} \right)$$

– in components by integrating out $\tilde{\theta}, \theta$,

$$\delta I = \int_{\Sigma_{\text{red}}} d\tilde{z}dz \left( \mu_{\tilde{z}}^{\tilde{z}} T_{zz} + \chi_{\tilde{z}}^{\theta} S_{z\theta} + \tilde{\mu}_{z}^{\tilde{z}} T_{\tilde{z}\tilde{z}} + \tilde{\chi}_{z}^{\tilde{\theta}} \tilde{S}_{\tilde{z}\tilde{\theta}} \right)$$

– recover Beltrami differentials $\mu, \tilde{\mu}$ and worldsheet gravitino fields $\chi, \tilde{\chi}$

$$H_{\tilde{\theta}}^{\tilde{z}} = \tilde{\theta}(\mu_{\tilde{z}}^{\tilde{z}} + \theta \chi_{\tilde{z}}^{\theta}) \quad \tilde{H}_{\theta}^{\tilde{z}} = \theta(\tilde{\mu}_{z}^{\tilde{z}} + \tilde{\theta} \tilde{\chi}_{z}^{\tilde{\theta}})$$

– Finite deformations of the metric with $\tilde{\mu} = \bar{\mu}$ and $\tilde{\chi} = \bar{\chi}$

integrate to the standard 2-dim $\mathcal{N} = 1$ supergravity action

(Brink, Di Vecchia, Howe; Deser, Zumino 1976)

• Type II superstring perturbation theory requires $\tilde{\mu} \neq \bar{\mu}$ and $\tilde{\chi} \neq \bar{\chi}$
Type II string amplitude

- Parametrize deformations $\tilde{H}_\theta \tilde{z}, H_\theta \tilde{z}$ by slice $\{\tilde{J}(\tilde{m}), J(m)\}$ in $\mathcal{M}_L \times \mathcal{M}_R$

  $H_\theta \tilde{z} = \tilde{D}_\theta V^z + H_A \delta m^A$
  $H_A = \partial \mathcal{J}_\theta \tilde{z} / \partial \tilde{m}^A$
  $m^A = (m^i, \zeta^\alpha)$

  $\tilde{H}_\theta \tilde{z} = D_\theta \tilde{V}^\tilde{z} + \tilde{H}_A \delta \tilde{m}^\tilde{A}$
  $\tilde{H}_A = \partial \mathcal{J}_\theta \tilde{z} / \partial \tilde{m}^\tilde{A}$
  $\tilde{m}^\tilde{A} = (\tilde{m}^i, \tilde{\zeta}^\alpha)$

  ghost fields

  $V^z \rightarrow C^z = c^z + \theta \gamma^\theta$
  $H_\theta \tilde{z} \rightarrow B_{z\theta} = \beta_{z\theta} + \theta b_{zz}$

  $V^\tilde{z} \rightarrow \tilde{C}^\tilde{z} = \tilde{c}^\tilde{z} + \tilde{\theta} \tilde{\gamma}^\tilde{\theta}$
  $H_\theta \tilde{z} \rightarrow \tilde{B}_{\tilde{z}\tilde{\theta}} = \tilde{\beta}_{\tilde{z}\tilde{\theta}} + \tilde{\theta} \tilde{b}_{\tilde{z}\tilde{z}}$

- Super conformal invariant ghost action

  $I_{gh} = \int_\Sigma [d\tilde{z}dz|d\tilde{\theta}d\theta] \left( B_{z\theta} \tilde{D}_\theta C^z + \tilde{B}_{\tilde{z}\tilde{\theta}} D_\theta C^\tilde{z} + B_{z\theta} H_A \delta m^A + \tilde{B}_{\tilde{z}\tilde{\theta}} \tilde{H}_A \delta \tilde{m}^\tilde{A} \right)$

- The integrand for the full amplitude is given by

  $\int D(XB\tilde{B}C\tilde{C}) \mathcal{V}_1 \cdots \mathcal{V}_n \prod_{\tilde{A},A} [d\tilde{m}^\tilde{A}dm^A] \delta(\langle \tilde{B}, \tilde{H}_\tilde{A} \rangle) \delta(\langle B, H_A \rangle) e^{-I_m - I_{gh}}$

  - $\mathcal{V}_1 \cdots \mathcal{V}_n$ are BRST-invariant vertex operators.
  - Picture Changing Operator formalism (Friedan, Martinec, Shenker 1986)
    - $\star$ may be obtained as singular limit for $\chi$ supported at points
    - $\star$ globally regular reformulation via “vertical integration” (Sen, Witten 2016)
Loop momenta and Chiral amplitudes

- $h$ independent loop momenta $p^\mu_I$ defined to flow across $\mathcal{A}_I$ cycles
  \[ p^\mu_I = \oint_{\mathcal{A}_I} dz \partial_z x^\mu \]

- Chiral Amplitudes (ED, Phong 1988)
  - Massless NS bosons with factorized polarization tensor $\tilde{\varepsilon}^\mu_i = \varepsilon^\mu_i \tilde{\varepsilon}_i$
  - Chiral amplitude at fixed loop momenta is given by
    \[ \mathcal{F}_R(\mathcal{J}, \varepsilon_i, k_i, p_I) = \left\langle \mathcal{V}_1 \cdots \mathcal{V}_N e^{\int d\tilde{x} \partial_\tilde{x} x^\mu} \right\rangle \prod_A \delta(\langle B, H_A \rangle) \mathrm{d}m^A \]
  - Correlation functions $\langle \cdots \rangle$ computed with chiral Green functions

- Full Superstring Amplitudes
  - obtained by pairing left and right and integrating over $\Gamma \in \mathcal{M}_L \times \mathcal{M}_R$
    \[ \mathcal{A}^{(h)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = \int_{\mathbb{R}^{10}} dp^\mu_I \int_{\Gamma} \mathcal{F}_L(\tilde{\mathcal{J}}, \tilde{\varepsilon}_i, k_i, p^\mu_I) \mathcal{F}_R(\mathcal{J}, \varepsilon_i, k_i, p_I^\mu) \]
  - integration over vertex operator insertion points included in integration over $\Gamma$
  - cfr “double copy construction” in supergravity calculations
Parametrization of super moduli

- **Superconformal structure** $\mathcal{J} \in \mathcal{M}_h$ specified by transition functions
  - Concrete calculations use parametrization by gravitino field $\chi \tilde{z}^\theta$

- **Local parametrization of moduli** (in conformal-invariant theory)
  - Conformal structure $\mathcal{J}$ with metric $g = |dz|^2$ in local coordinates $(z, \tilde{z})$
  - deform conformal structure by Beltrami differential to $g' = |dz + \mu d\tilde{z}|^2$
  - realized in CFT by inserting $\int_{\Sigma} d\tilde{z} dz \mu \tilde{z} T_{zz}$ to all orders in $\mu$

- **Local parametrization of supermoduli** (in superconformal-invariant theory)
  - Start with $\Sigma_{\text{red}}$ with complex structure given by $\mathcal{J} \in \mathcal{M}_{\text{red}}$
  - Deform super conformal structure by inserting $T$ and $S$

  $$\int_{\Sigma_{\text{red}}} d\tilde{z} dz \left( \mu \tilde{z} T_{zz} + \chi \tilde{z}^\theta S_{z\theta} \right)$$

  - $\chi$ and $\mu$ parametrized by local odd coordinates on $\mathcal{M}_h$

- **For $h = 2$, even spin structures, holó projection** $\mathcal{M}_2 \to \mathcal{M}_2$ exists
  - via the super period matrix (ED, Phong 2001)

- **For $h \geq 5$ no holó projection** $\mathcal{M}_h \to \mathcal{M}_h$ exists (Donagi, Witten 2013)
The super period matrix (even spin structures)

- Start from conformal structure $J$ for $\Sigma_{\text{red}}$ with holó 1-forms $\omega_I$
  \[
  \oint_{A_I} \omega_J = \delta_{IJ} \quad \oint_{B_I} \omega_J = \Omega_{IJ} \quad I, J = 1, 2
  \]

- Deform to superconformal structure $\mathcal{J}$ on $\Sigma$ with superholó forms $\hat{\omega}_I$
  \[
  \oint_{A_I} \hat{\omega}_J = \delta_{IJ} \quad \oint_{B_I} \hat{\omega}_J = \hat{\Omega}_{IJ} \quad I, J = 1, 2
  \]

  - Explicit formula for the super period matrix $\hat{\Omega}$ for even spin structure $\delta$
    \[
    \hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int_{\Sigma_{\text{red}}} \omega_I(z) \chi(z) S_\delta(z, w|\Omega) \chi(w) \omega_J(w) + \int_{\Sigma_{\text{red}}} \mu \omega_I \omega_J
    \]

  - $\hat{\Omega}_{IJ}$ is locally supersymmetric; $\hat{\Omega}_{IJ} = \hat{\Omega}_{JI}$; and $\text{Im} \hat{\Omega} > 0$
  - Every $\hat{\Omega}$ corresponds to an ordinary Riemann surface
  - Szegö kernel $S_\delta(z, w|\Omega)$ is non-singular in the interior of $\mathcal{M}_2$

$\Rightarrow$ Projection using $\hat{\Omega}$ is holomorphic and natural for genus 2
Projecting and pairing Chiral Amplitudes

- **Chiral Amplitudes on $\mathcal{M}_2$**
  - Natural parametrization of $\mathcal{M}_2$ by $(\hat{\Omega}_{IJ}, \zeta^\alpha)$ (even spin structure $\delta$)
  - involves measure $d\kappa[\delta](\hat{\Omega}, \zeta)$ and correlation functions $C[\delta](\epsilon_i, k_i, p_I|\hat{\Omega}, \zeta)$

- **Projection to chiral amplitudes on $\mathcal{M}_2$**
  - by integrating over $\zeta$ and summing over $\delta$ at fixed $\hat{\Omega}$
    $\mathcal{R}(\epsilon_i, k_i, p_I|\hat{\Omega}) = \sum_\delta \int_\zeta d\kappa[\delta](\hat{\Omega}, \zeta) C[\delta](\epsilon_i, k_i, p_I|\hat{\Omega}, \zeta)$
    $\mathcal{L}(\bar{\epsilon}_i, k_i, p_I|\hat{\Omega}) = \sum_\tilde{\delta} \int_{\tilde{\zeta}} d\kappa[\tilde{\delta}](\hat{\Omega}, \tilde{\zeta}) C[\tilde{\delta}](\bar{\epsilon}_i, k_i, p_I|\hat{\Omega}, \tilde{\zeta})$
  - for heterotic, $\mathcal{L}$ is chiral half of bosonic string, has no integral in $\tilde{\zeta}$
  - phase factors determined by $Sp(4, \mathbb{Z})$ modular invariance

- **Pairing left and right chiral amplitudes, integrating over $p_I$ and $\hat{\Omega}$**
  $A^{(2)}(\epsilon_i, \bar{\epsilon}_i, k_i) = \int_{\mathcal{M}_2} d\hat{\Omega} \int dp^\mu_I \mathcal{R}(\epsilon_i, k_i, p_I|\hat{\Omega}) \mathcal{L}(\bar{\epsilon}_i, k_i, p_I|\hat{\Omega})$
  - Integral over $p_I$ is Gaussian and can be carried out explicitly.
Genus two

• Siegel Upper half space $S_2$

\[ S_2 = \{ \Omega_{IJ} = \Omega_{JI} \in \mathbb{C} \text{ with } I, J = 1, 2 \text{ and } Y = \text{Im}\Omega > 0 \} \]

- $Sp(4, \mathbb{R})$ acts by $\Omega \rightarrow (A\Omega + B)(C\Omega + D)^{-1}$

\[ M^t J M = J \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \]

- $S_2$ has $Sp(4, \mathbb{R})$-invariant metric $ds_2^2$ and volume form $d\mu_2$

\[ ds_2^2 = \sum_{I,J,K,L=1,2} Y_{IJ}^{-1} d\bar{\Omega}_{JK} Y_{KL}^{-1} d\Omega_{LI} \]

• Compact Riemann surfaces $\Sigma$

- Choose canonical homology basis of $\mathcal{A}_I, \mathcal{B}_I$ cycles for $H_1(\Sigma, \mathbb{Z})$.
- $\omega_I$ dual holomorphic $(1,0)$ forms,

\[ \oint_{\mathcal{A}_I} \omega_J = \delta_{IJ} \quad \oint_{\mathcal{B}_I} \omega_J = \Omega_{IJ} \]

- Riemann relations imply $\Omega \in S_2$;
- Modular group $Sp(4, \mathbb{Z})$; moduli space $M_2 = S_2/Sp(4, \mathbb{Z})$. 
Genus-two Type II four-graviton amplitude

- **Type II four-graviton amplitude**  (ED, Phong 2001 – 2005)

\[
A^{(2)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = g_s^2 \mathcal{K} \mathcal{\tilde{K}} \int_{\mathcal{M}_2} d\mu_2 \mathcal{B}^{(2)}(s_{ij} | \Omega)
\]

\[
\mathcal{B}^{(2)}(s_{ij} | \Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \mathcal{\tilde{Y}}}{(\det \text{Im} \Omega)^2} \exp \left( \sum_{i<j} s_{ij} G(z_i, z_j | \Omega) \right)
\]

- \(G(z_i, z_j)\) is the genus-two scalar Green function;
- \(\Delta(z_i, z_j)\) is a bi-holomorphic form independent of \(s, t, u\).

\[
\Delta(z, w) = \omega_1(z) \wedge \omega_2(w) - \omega_2(z) \wedge \omega_1(w)
\]

\[
\mathcal{Y} = (t - u) \Delta(z_1, z_2) \wedge \Delta(z_3, z_4) + (s - t) \Delta(z_1, z_3) \wedge \Delta(z_4, z_2)
\]

\[
+(u - s) \Delta(z_1, z_4) \wedge \Delta(z_2, z_3)
\]

- reproduced (with fermions) in pure spinor formulation  (Berkovits, Mafra 2005)

- **Singularity structure**
  - For fixed \(\Omega\) integrations over \(\Sigma\) produce poles in \(\mathcal{B}\) at positive integers \(s_{ij}\).
  - The integral over \(\Omega\) requires analytic continuation beyond \(\text{Re}(s_{ij}) = 0\).
  - Branch cuts in \(s_{ij}\) starting at integers produced from \(\Omega_{11}, \Omega_{22} \rightarrow i\infty\)
Genus-two Heterotic four-graviton amplitude

• Heterotic four NS boson amplitude at genus 2 (ED, Phong 2005)

\[ A^{(2)}_{\mathcal{O}}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = g_s^2 \kappa \int_{M_2} d\mu_2 B^{(2)}_{\mathcal{O}}(\tilde{\varepsilon}_i, k_i|\Omega) \]

\[ B^{(2)}_{\mathcal{O}}(\tilde{\varepsilon}_i, k_i|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \mathcal{W}_{\mathcal{O}}(\tilde{\varepsilon}_i, k_i)}{(\det \text{Im}\Omega)^2 \Psi_{10}(\Omega)} \exp \left( \sum_{i<j} s_{ij} G(z_i, z_j) \right) \]

– \( \Psi_{10}(\Omega) \) is the Igusa cusp form.

• Dependence of the operator \( \mathcal{O} \) on the channel:
  * 4 gravitons \( \mathcal{R}^4 \)
  * 2 gravitons + 2 gauge bosons \( \mathcal{R}^2 \text{tr}(\mathcal{F}^2) \)
  * 4 gauge bosons \( (\text{tr}\mathcal{F}^2)^2 \)
  * 4 gauge bosons \( \text{tr}(\mathcal{F}^4) \)

– For example,

\[ \mathcal{W}_{\mathcal{R}^4}(\tilde{\varepsilon}_i, k_i) = \frac{\langle \prod_{i=1}^4 \tilde{\varepsilon}_i \cdot \bar{\partial} \tilde{x}(z_i) \ e^{ik_i \cdot \tilde{x}(z_i)} \rangle}{\langle \prod_{i=1}^4 e^{ik_i \cdot \tilde{x}(z_i)} \rangle} \]

– Gauge parts are obtained by the correlators of the current \( (0, 1) \)-forms.
Singularities in the projection $\overline{M}_2 \to \overline{M}_2$

- Projection $\overline{M}_2 \to \overline{M}_2$ is holó, but integration extends to boundary
  - are there singularities in the projection $\overline{M}_2 \to \overline{M}_2$?

  $\Omega = \begin{pmatrix} \tau & u \\ u & \sigma \end{pmatrix}$

  $u \to 0$ separating node  
  $\sigma \to i\infty$ non-separating node

- Key ingredient in $\hat{\Omega}$ is the Szegö kernel

  $S_\delta(z, w|\Omega) = \frac{\vartheta[\delta](z - w|\Omega)}{\vartheta[\delta](0|\Omega) E(z, w)}$

  - As $u \to 0$ we have $\vartheta[\delta](0|\Omega) \to \vartheta[\delta_1](0|\tau) \vartheta[\delta_2](0|\tau)$
  - Even $\delta = [\delta_1, \delta_2]$ with $\delta_1, \delta_2$ odd produces a singularity in $S_\delta$ and $\hat{\Omega}$

- Physical effects
  - singularity killed by $\psi$-zero modes in $\mathbb{R}^{10}$ (space-time susy)
  - contribution when susy is broken by radiative corrections (Witten 2013)
  - Two-loop vacuum energy in Heterotic strings on CY orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$
    $\star$ is zero for $E_8 \times E_8 \to E_6 \times E_8$ with unbroken susy
    $\star$ non-zero for $\text{Spin}(32)/\mathbb{Z}_2 \to SO(26) \times U(1)$ with broken susy
  (Atick, Dixon, Sen 1988; Dine, Seiberg, Witten; ED, Phong 2013; Berkovits, Witten 2014)
Singularities in the projection $\mathcal{M}_3 \rightarrow \mathcal{M}_3$

• Some basic structure theorems
  – A hyper-elliptic surface is a branched double cover of the sphere $S^2$;
  – All genus 1 and all genus 2 surfaces are hyper-elliptic;
  – Hyper-elliptic surfaces form a co-dim 1 sub-variety in the interior of $\mathcal{M}_3$
    (referred to as the hyper-elliptic divisor)

• The genus-three period matrix (for even spin structure)

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int \int \omega_I(z)\chi(z)S_\delta(z,w|\Omega)\chi(w)\omega_J(w) + O(\chi^4)$$

  – For $\Omega$ on the hyper-elliptic divisor of $\mathcal{M}_3$
    there always exists an even spin structure $\delta$ such that $\vartheta[\delta](0|\Omega) = 0$
  – the presence of the extra Dirac zero modes kills effects of this singularity

⇒ Beautiful proposal for the genus 3 superstring measure
  (Cacciatori, Dalla Piazza, van Geemen 2008)

  – Another even $\delta$ does produce a subtle singularity in $\hat{\Omega}$ (Witten 2015)