Lectures on Superstring Amplitudes

Part 2: Superstrings

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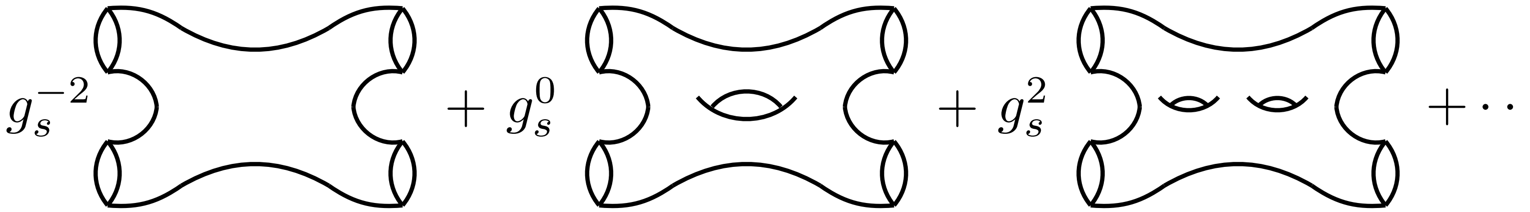
Superstring Perturbation Theory

- **Theory of fluctuating random surfaces** (closed strings shown)
  - governed by topological expansion in the genus h weighed by g_s^{2h-2}



- **Bosonic string**
  - unstable with closed string tachyon
  - Nature has fermions !

- **Superstrings generalize bosonic string**
  - they have fermions
  - no tachyon
  - supersymmetry

Approaches to Superstring Perturbation Theory

- **Goal is to obtain superstring amplitudes at all genera**

  – Ramond-Neveu-Schwarz formulation of fermionic strings;
  w/ Gliozzi-Scherk-Olive projection to supersymmetric spectrum;
  – Green-Schwarz space-time supersymmetric formulation;
  – Mandelstam light-cone formulation;
  – String field theory;
  – Topological string theory;
  – Berkovits pure spinor formulation.

- **Different perturbative superstring theories** (in 10 dimensions)

  – Type I open & closed, orientable & non-orientable, D-branes
  – Type IIA,B closed orientable, D-branes
  – Heterotic closed orientable $E_8 \times E_8$, $Spin(32/\mathbb{Z}_2)$

- Here: **RNS formulation, closed orientable superstrings, dimension 10**

Genus-zero four-graviton superstring amplitude

- **Kinematics of the four-graviton amplitude**
  - momenta of gravitons k_i^μ are conserved $\sum_i k_i^\mu = 0$
  - choose basis of factorized polarization tensors $\varepsilon_i^{\mu\nu} = \varepsilon_i^\mu \tilde\varepsilon_i^\nu$
  - masslessness $k_i^2 = 0$ and transversality $k_i^\mu \varepsilon_i^\mu = k_i^\mu \tilde\varepsilon_i^\mu = 0$ for $i = 1,2,3,4$
  - kinematic invariants $s = s_{12} = s_{34}$, $t = s_{14} = s_{23}$, $u = s_{13} = s_{24}$

$$s_{ij} = -\alpha'(k_i + k_j)^2/4$$

- **Tree-level four-graviton amplitude is given by**

$$\mathcal{A}^{(0)}(\varepsilon_i, \tilde\varepsilon_i, k_i) = \frac{1}{g_s^2} \times \mathcal{K}\tilde{\mathcal{K}} \times \frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)}$$

  - Kinematical factor $\mathcal{K}$ given in terms of $f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - k_i^\nu \varepsilon_i^\mu$ by

$$\begin{aligned} \mathcal{K} \quad = \quad & (f_1 f_2)(f_3 f_4) + (f_1 f_3)(f_2 f_4) + (f_1 f_4)(f_2 f_3) \\ & -4(f_1 f_2 f_3 f_4) - 4(f_1 f_2 f_4 f_3) - 4(f_1 f_3 f_2 f_4) \end{aligned}$$

  - for $\tilde{\mathcal{K}}$ replace ε_i by $\tilde\varepsilon_i$
  - Equivalently, $\mathcal{K} \times \tilde{\mathcal{K}} = \mathcal{R}^4$ with $\mathcal{R}$ the linearized Weyl tensor
  - String duality: symmetric in s, t, u
  - Poles in each channel, at $s, t, u = 0, 1, 2, \cdots$

Genus-one four-graviton superstring amplitude

- **Type II four-graviton amplitude to one-loop order** (Green, Schwarz 1982)

$$\mathcal{A}^{(1)}(\varepsilon_i, \tilde\varepsilon_i, k_i) = \mathcal{R}^4 \int_{\mathcal{M}_1} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau)$$

  – Partial amplitude $\mathcal{B}^{(1)}$ is a modular function in $\tau \in \mathcal{M}_1 = \mathcal{H}_1/SL(2,\mathbb{Z})$

$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \int_{\Sigma^4} \prod_{i=1}^{4} \frac{d^2 z_i}{\mathrm{Im}\,\tau} \exp\Big(\sum_{i<j} s_{ij}\, G(z_i - z_j|\tau) \Big)$$

  – $G(z|\tau)$ is the scalar Green function on the torus Σ of modulus τ.
  – Analogous formulas for Heterotic strings and more external states.

- **Singularity structure**
  – For fixed τ integrations over Σ produce poles in $\mathcal{B}^{(1)}$ at positive integers s_{ij}.
  – The integral over τ converges absolutely only for $\mathrm{Re}(s_{ij}) = 0$.
  – Analytic continuation to $s_{ij} \in \mathbb{C}$ via decomposition of $\mathcal{M}_1$.
  – Branch cuts in s_{ij} starting at integers ≥ 0 are produced by $\tau \to i\infty$ region.

Loop momenta

- **Loop momenta may be exposed**
  - Choose a canonical basis of homology cycles $\mathfrak{A}, \mathfrak{B}$.
  - Choose loop momentum p flowing through the cycle $\mathfrak{A}$,

$$\int_{\mathcal{M}_1} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau) = \int_{\mathbb{R}^{10}} d^{10}p \int_{\mathcal{M}_1} \int_{\Sigma^4} \Big| \mathcal{F}(z_i, k_i, p|\tau) \Big|^2$$

- **Chiral amplitude $\mathcal{F}$ is locally holomorphic in τ and z_i**

$$\mathcal{F}(z_i, k_i, p|\tau) = e^{i\pi\tau p^2 + 2\pi i p \sum_i k_i z_i} \prod_{i<j} \vartheta_1(z_i - z_j|\tau)^{-s_{ij}} \, d\tau \prod_{i=1}^{4} dz_i$$

  - at the cost of non-trivial monodromy

$$\begin{aligned} \mathcal{F}(z_i + \delta_{i,\ell}\mathfrak{A}, k_i, p|\tau) &= e^{2\pi i k_\ell \cdot p} \mathcal{F}(z_i, k_i, p|\tau) \\ \mathcal{F}(z_i + \delta_{i,\ell}\mathfrak{B}, k_i, p|\tau) &= \mathcal{F}(z_i, k_i, p + k_\ell|\tau) \end{aligned}$$

  - Modular invariance of $\mathcal{A}^{(1)}$ guarantees independence of choices.
  - Hermitian pairing of $\mathcal{F}$ and $\bar{\mathcal{F}}$ is familiar from 2-d CFT where loop momentum p labels conformal blocks of 10 copies of $c = 1$.

UV-finiteness

- **Thanks to modular invariance, all string amplitudes are UV-finite**
  - shown for the closed bosonic string at genus one (Shapiro 1972)
  - holds for all modular invariant superstrings to all loops (i.e. all genera)

- **For genus-one: All chiral amplitudes have a universal factor**

$$\mathcal{F}(z_i, \varepsilon_i, k_i, p_I|\tau) = e^{i\pi p^\mu \tau p^\mu} \times \cdots$$

  - Modular invariance allows one to choose
    a fundamental domain where $\mathrm{Im}(\tau)$ bounded from below

$$\mathcal{H}_1/SL(2,\mathbb{Z}) = \{\tau \in \mathbb{C}, \mathrm{Im}(\tau) > 0, \, |\tau| \geq 1, \, |\mathrm{Re}(\tau)| \leq \frac{1}{2}\}$$

  - Analogous, more complicated, choices to higher genus

$\Rightarrow$ **Uniform Gaussian suppression at large loop momenta**

$\Rightarrow$ **UV finiteness**

RNS formulation of superstrings

- $M = \mathbb{R}^{10}$ **flat Minkowski space-time with Lorentz group** $SO(1,9)$
  - x^μ scalars on worldsheet Σ, map Σ into M
  - ψ^μ spinors on Σ but Lorentz vector under $SO(1,9)$
    - ⋆ Worldsheet supersymmetry $\Longrightarrow$ Σ is a super Riemann surface
    - ⋆ Two sectors : NS bosons $SO(1,9)$-tensors
      R fermions $SO(1,9)$-spinors

- **With Minkowski signature** Σ
  - ψ^μ and $\tilde\psi^\mu$ are *independent* Majorana-Weyl spinors of opposite chirality

- **With Euclidean signature** Σ
  - ψ^μ and $\tilde\psi^\mu$ must be *independent* complex Weyl spinors
  - Globally, on a compact Riemann surface of genus h,
    - ⋆ All ψ^μ are sections of a the same spin bundle S (and $\tilde\psi^\mu$ of $\tilde S$)
    - ⋆ 2^{2h} distinct spin structures for S (and 2^{2h} independently for $\tilde S$)

- **GSO projection requires independent summation over spin structures**

Quantization of worldsheet spinor fields

- **Illustrate**
  - Ramond and Neveu-Schwarz sectors
  - independence of chiralities

- **Dirac action and equation for flat $M = \mathbb{R}^{10}$ with metric η**
  - All components of ψ_+^μ are sections of the same spin bundle S
  - Complex structure J with local complex coordinates $(z, \tilde{z})$
  - Dirac action,

$$I_\psi[\psi, J] = \frac{1}{2\pi} \int_\Sigma d\tilde{z} dz \, \psi_+^\mu \partial_{\tilde{z}} \psi_+^\nu \eta_{\mu\nu}$$

  - Dirac equation $\partial_{\tilde{z}} \psi_+^\mu = 0$ has locally holomorphic solutions,
  - but products of operators produce singularities

$$\psi_+^\mu(z)\psi_+^\nu(w) = \frac{\eta^{\mu\nu}}{z-w} + \text{regular}$$

  - each component ψ^μ generates a CFT with central charge $c = \frac{1}{2}$.

Quantization of worldsheet spinor fields (cont'd)

- **Quantization on flat cylinder or conformal equivalent flat annulus**
  - cylinder $w = \tau + i\sigma$ with identification $\sigma \approx \sigma + 2\pi$
  - annulus centered at $z = 0$, conformally mapped by $z = e^w$
  - one-forms related by $dz = e^w \, dw$, spinors by $(dz)^{\frac{1}{2}} = e^{w/2} \, (dw)^{\frac{1}{2}}$
  - fields related by conformal transformation $\psi_{\rm cyl}(z) = e^{w/2} \, \psi_{\rm ann}(w)$

- **Two possible spin structures**

NS $\quad \psi^\mu_{\rm cyl}(\tau, \sigma + 2\pi) = - \psi^\mu_{\rm cyl}(\tau, \sigma) \quad$ or $\quad \psi^\mu_{\rm ann}(e^{2\pi i} \, z) = + \psi^\mu_{\rm ann}(z)$

R $\quad \psi^\mu_{\rm cyl}(\tau, \sigma + 2\pi) = + \psi^\mu_{\rm cyl}(\tau, \sigma) \quad$ or $\quad \psi^\mu_{\rm ann}(e^{2\pi i} \, z) = - \psi^\mu_{\rm ann}(z)$

- **Free field quantization in annulus representation**

NS $\quad \psi^\mu(z) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} b^\mu_r \, z^{-\frac{1}{2} - r} \qquad \{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu} \delta_{r+s,0}$

R $\quad \psi^\mu(z) = \sum_{n \in \mathbb{Z}} d^\mu_n \, z^{-\frac{1}{2} - n} \qquad \{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu} \delta_{m+n,0}$

Quantization of worldsheet spinor fields (cont'd)

- **Lorentz generators of $SO(1,9)$:** $[J^{\mu\nu}, \psi^\kappa(z)] = \eta^{\nu\kappa}\psi^\mu(z) - \eta^{\mu\kappa}\psi^\nu(z)$

$$J^{\mu\nu}_{\rm NS} = \sum_{r \in \mathbb{N}-\frac{1}{2}} \left(b^\mu_{-r} b^\nu_r - b^\nu_{-r} b^\mu_r \right)$$

$$J^{\mu\nu}_{\rm R} = \frac{1}{2}[d^\mu_0, d^\nu_0] + \sum_{n \in \mathbb{N}} \left(d^\mu_{-n} d^\nu_n - d^\nu_{-n} d^\mu_n \right)$$

- **Fock space construction produces two sectors**

  ⋆ NS ground state defined by $b^\mu_r|0;{\rm NS}\rangle = 0$ for all $r > 0$
  - $|0;{\rm NS}\rangle$ is unique and in trivial representation of $SO(1,9)$
  - Fock space = linear combinations of $b^{\mu_1}_{-r_1} \cdots b^{\mu_p}_{-r_p}|0;{\rm NS}\rangle$, $r_i > 0$
  - All states in tensor reps of $SO(1,9)$ are space-time bosons.

  ⋆ R ground state defined by $d^\mu_n|0,\alpha;{\rm R}\rangle = 0$ for all $n > 0$
  - $|0,\alpha;{\rm R}\rangle$ is degenerate and in spinor rep. of $SO(1,9)$, states labelled by α
  - Fock space = linear combinations of $d^{\mu_1}_{-n_1} \cdots d^{\mu_p}_{-n_p}|0,\alpha;{\rm R}\rangle$, $n_i > 0$
  - All states in spinor reps of $SO(1,9)$ are space-time fermions.

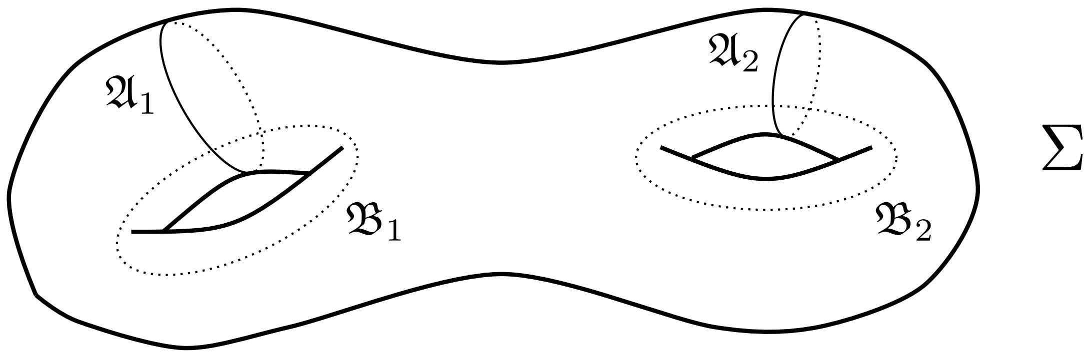
Summation over spin structures

- **Theory with bosons and fermions requires both NS and R sectors**
  - to include both, one must sum over two spin structures of the annulus
- **Type II spin structures of $\psi^\mu_\pm$ are independent of one another**
  - space-time fermions are in the R $\otimes$ NS and NS $\otimes$ R sectors
    which could never arise if spin structures for opposite chiralities coincided
- **On the torus, viewed as cylinder + identification**
  - spin structures along cycle of cylinder produce R and NS sectors
  - sum over spin structures along conjugate cycle produces GSO-projection
    - ⋆ reduces to half the states in both R and NS sectors
    - ⋆ R-sector: space-time spinor of definite chirality
    - ⋆ NS-sector: eliminates the tachyon

  $\Rightarrow$ sum over *all* spin structures

Summation over spin structures (cont'd)

- **Fix a canonical homology basis of cycles $\mathfrak{A}_I, \mathfrak{B}_I$ of $H_1(\Sigma, \mathbb{Z})$ $I = 1, \cdots, h$**
  - with canonical intersection pairing
    $\#(\mathfrak{A}_I, \mathfrak{A}_J) = \#(\mathfrak{B}_I, \mathfrak{B}_J) = 0$ and $\#(\mathfrak{A}_I, \mathfrak{A}_J) = \delta_{IJ}$



- **Transformations which maps one canonical basis into another**
  - linear with integer coefficients
  - preserve the intersection matrix: $Sp(2h, \mathbb{Z})$

- **On Riemann surface of higher genus h sum over all spin structures**
  - along $\mathfrak{A}$-cycles produces R and NS sectors
  - along $\mathfrak{B}$-cycles produces GSO-projection
  - mapped into one another by $Sp(2h, \mathbb{Z}_2)$

Super Riemann surfaces

- **Ordinary Riemann surface** (locally $\mathbb{C}$ with coordinate z)
  - complex manifold: holomorphic transition functions $z \to z'(z)$;
  - complex structure = conformal structure J
  - Moduli space $\mathcal{M}_h = \{J\}/\mathrm{Diff}(\Sigma)$ of genus h compact Riemann surfaces

- **Complex super manifold** (locally $\mathbb{C}^{1|1}$ with coordinates $z|\theta$)
  - holó transition functions $z|\theta \to z'(z,\theta)|\theta'(z,\theta)$ generate $\mathcal{N}=2$ super conformal

- **Super Riemann surface** (locally $\mathbb{C}^{1|1}$ with coordinates $z|\theta$)
  - holó transition functions $z|\theta \to z'|\theta'$ rescale $D_\theta = \partial_\theta + \theta\partial_z$
  - Transition functions define $\mathcal{N}=1$ superconformal structure $\mathcal{J}$
  - Globally: $T\Sigma$ has a completely non-integrable subbundle of rank $0|1$

- **Moduli space of compact super Riemann surfaces**: $\mathfrak{M}_h = \{\mathcal{J}\}/\mathrm{Diff}(\Sigma)$
  = equivalence classes of superconformal structures $\mathcal{J}$

$$\dim_{\mathbb{C}} \mathfrak{M}_h = \begin{cases} 0|0 & h=0 \\ 1|0 \text{ or } 1|1 & h=1 \quad \text{even or odd spin structure} \\ 3h-3|2h-2 & h \geq 2 \end{cases}$$

  - odd modulus at $h=1$ odd spin structure is a book keeping device;
  - odd moduli really first appear at genus 2, as curved super spaces.

Superstring worldsheets and moduli spaces

- **Heterotic**
  - Left : RS Σ_L, moduli space $\mathcal{M}_L$ coord resp. $\tilde{z}$ and $\tilde{m}^i$
  - Right : SRS Σ_R, moduli space $\mathfrak{M}_R$ coord resp. (z,θ) and (m^i,ζ^α)
  - Worldsheet is a cycle $\Sigma \subset \Sigma_L \times \Sigma_R$ of dim $1|1$

    subject to $\Sigma_{\rm red} = {\rm diag}(\Sigma_{L\,{\rm red}} \times \Sigma_{R\,{\rm red}})$: $\tilde{z}^* = z + {\rm nilpotent}$
  - Moduli space is a cycle $\Gamma \subset \mathcal{M}_L \times \mathfrak{M}_R$ of dim $3h-3|2h-2$ for $h \geq 2$

    subject to $\Gamma_{\rm red} = {\rm diag}(\mathcal{M}_{L\,{\rm red}} \times \mathfrak{M}_{R\,{\rm red}})$: $(\tilde{m}^i)^* = m^i + {\rm nilpotent}$

    (reduced space obtained by setting all nilpotent variables to zero)

- **Type II**
  - Left : SRS Σ_L, moduli space $\mathfrak{M}_L$ coord resp. $(\tilde{z},\tilde{\theta})$ and $(\tilde{m}^i,\tilde{\zeta}^\alpha)$
  - Right : SRS Σ_R, moduli space $\mathfrak{M}_R$ coord resp. (z,θ) and (m^i,ζ^α)
  - Worldsheet is a cycle $\Sigma \subset \Sigma_L \times \Sigma_R$ of dim $1|2$
  - Moduli space is cycle $\Gamma \subset \mathfrak{M}_L \times \mathfrak{M}_R$ of dim $3h-3|4h-4$ for $h \geq 2$

    subject to $\tilde{z}^* = z + {\rm nilpotent}$ and $(\tilde{m}^i)^* = m^i + {\rm nilpotent}$

- **Super-Stokes theorem ensures independence of the choice of cycles**
  - in amplitudes with BRST invariant vertex operators
  - consistent definition of superstring amplitudes to all genera (Witten 2012)

Worldsheet action for Type II superstrings

- **Worldsheet is** $\Sigma \subset \Sigma_L \times \Sigma_R$
  - Σ_L has superconformal structure $\tilde{\mathcal{J}}$ with local coordinates $\tilde{z}|\tilde{\theta}$
  - Σ_R has superconformal structure $\mathcal{J}$ with local coordinates $z|\theta$

- **Superconformal invariant matter action**
  - worldsheet matter field

$$X^\mu(\tilde{z}, z|\tilde{\theta}, \theta) = x^\mu(\tilde{z}, z) + \theta\psi^\mu(\tilde{z}, z) + \tilde{\theta}\tilde{\psi}^\mu(\tilde{z}, z) + \tilde{\theta}\theta F^\mu(\tilde{z}, z)$$

  - Worldsheet action in local coordinates $(D_\theta = \partial_\theta + \theta\partial_z)$

$$I_m[X^\mu, \tilde{\mathcal{J}}, \mathcal{J}] = \int_\Sigma [d\tilde{z}dz|d\tilde{\theta}d\theta]\tilde{D}_{\tilde{\theta}} X^\mu D_\theta X_\mu$$

  - Superconformal algebra on fields generated by

$$\mathcal{S}_{z\theta} = S_{z\theta} + \theta T_{zz} \qquad S_{z\theta} = \frac{1}{2}\psi^\mu \partial_z x_\mu \qquad T_{zz} = -\frac{1}{2}\partial_z x^\mu \partial_z x_\mu + \frac{1}{2}\psi^\mu \partial_z \psi_\mu$$

$$\tilde{\mathcal{S}}_{\tilde{z}\tilde{\theta}} = \tilde{S}_{\tilde{z}\tilde{\theta}} + \tilde{\theta}\tilde{T}_{\tilde{z}\tilde{z}} \qquad \tilde{S}_{\tilde{z}\tilde{\theta}} = \frac{1}{2}\tilde{\psi}^\mu \partial_{\tilde{z}} x_\mu \qquad \tilde{T}_{\tilde{z}\tilde{z}} = -\frac{1}{2}\partial_{\tilde{z}} x^\mu \partial_{\tilde{z}} x_\mu + \frac{1}{2}\tilde{\psi}^\mu \partial_{\tilde{z}} \tilde{\psi}_\mu$$

Deformations of superconformal structures

- **Under deformation of $\tilde{\mathcal{J}}$ for Σ_L and $\mathcal{J}$ for Σ_R**

$$\delta I = \int_\Sigma [d\tilde{z} dz | d\tilde{\theta} d\theta] \Big(H_{\tilde{\theta}}{}^z \mathcal{S}_{z\theta} + \tilde{H}_\theta{}^{\tilde{z}} \, \tilde{\mathcal{S}}_{\tilde{z}\tilde{\theta}} \Big)$$

  – in components by integrating out $\tilde{\theta}, \theta$,

$$\delta I = \int_{\Sigma_{\rm red}} d\tilde{z} dz \, \Big(\mu_{\tilde{z}}{}^z \, T_{zz} + \chi_{\tilde{z}}{}^\theta \, S_{z\theta} + \tilde{\mu}_z{}^{\tilde{z}} \, T_{\tilde{z}\tilde{z}} + \tilde{\chi}_z{}^{\tilde{\theta}} \, \tilde{S}_{\tilde{z}\tilde{\theta}} \Big)$$

  – recover Beltrami differentials $\mu, \tilde{\mu}$ and worldsheet gravitino fields $\chi, \tilde{\chi}$

$$H_{\tilde{\theta}}{}^z = \tilde{\theta}(\mu_{\tilde{z}}{}^z + \theta \chi_{\tilde{z}}{}^\theta) \qquad\qquad \tilde{H}_\theta{}^{\tilde{z}} = \theta(\tilde{\mu}_z{}^{\tilde{z}} + \tilde{\theta}\tilde{\chi}_z{}^{\tilde{\theta}})$$

  – Finite deformations of the metric with $\tilde{\mu} = \bar{\mu}$ and $\tilde{\chi} = \bar{\chi}$
    integrate to the standard 2-dim $\mathcal{N} = 1$ supergravity action
    (Brink, Di Vecchia, Howe; Deser, Zumino 1976)

- **Type II superstring perturbation theory requires $\tilde{\mu} \neq \bar{\mu}$ and $\tilde{\chi} \neq \bar{\chi}$**

Type II string amplitude

- **Parametrize deformations $\tilde{H}_\theta{}^{\tilde{z}}, H_{\tilde{\theta}}{}^z$ by slice $\{\tilde{\mathcal{J}}(\tilde{\mathfrak{m}}), \mathcal{J}(\mathfrak{m})\}$ in $\mathfrak{M}_L \times \mathfrak{M}_R$**

$$H_{\tilde{\theta}}{}^z = \tilde{D}_{\tilde{\theta}} V^z + H_A \delta \mathfrak{m}^A \qquad H_A = \partial \mathcal{J}_{\tilde{\theta}}{}^z / \partial \mathfrak{m}^A \qquad \mathfrak{m}^A = (m^i, \zeta^\alpha)$$

$$\tilde{H}_\theta{}^{\tilde{z}} = D_\theta \tilde{V}^{\tilde{z}} + \tilde{H}_{\tilde{A}} \delta \tilde{\mathfrak{m}}^{\tilde{A}} \qquad \tilde{H}_{\tilde{A}} = \partial \mathcal{J}_\theta{}^{\tilde{z}} / \partial \tilde{\mathfrak{m}}^{\tilde{A}} \qquad \tilde{\mathfrak{m}}^A = (\tilde{m}^i, \tilde{\zeta}^\alpha)$$

ghost fields

$$V^z \to C^z = c^z + \theta \gamma^\theta \qquad H_{\tilde{\theta}}{}^z \to B_{z\theta} = \beta_{z\theta} + \theta b_{zz}$$

$$V^{\tilde{z}} \to \tilde{C}^{\tilde{z}} = \tilde{c}^{\tilde{z}} + \tilde{\theta} \tilde{\gamma}^{\tilde{\theta}} \qquad H_\theta{}^{\tilde{z}} \to \tilde{B}_{\tilde{z}\tilde{\theta}} = \tilde{\beta}_{\tilde{z}\tilde{\theta}} + \tilde{\theta} \tilde{b}_{\tilde{z}\tilde{z}}$$

  - Super conformal invariant ghost action

$$I_{\rm gh} = \int_\Sigma [d\tilde{z} dz | d\tilde{\theta} d\theta] \Big(B_{z\theta} \tilde{D}_{\tilde{\theta}} C^z + \tilde{B}_{\tilde{z}\tilde{\theta}} D_\theta C^{\tilde{z}} + B_{z\theta} H_A \delta \mathfrak{m}^A + \tilde{B}_{\tilde{z}\tilde{\theta}} \tilde{H}_{\tilde{A}} \delta \tilde{\mathfrak{m}}^{\tilde{A}} \Big)$$

- **The integrand for the full amplitude is given by**

$$\int D(XB\tilde{B}C\tilde{C})\, \mathcal{V}_1 \cdots \mathcal{V}_n \prod_{\tilde{A},A} [d\tilde{\mathfrak{m}}^{\tilde{A}} d\mathfrak{m}^A]\, \delta(\langle \tilde{B}, \tilde{H}_{\tilde{A}} \rangle) \delta(\langle B, H_A \rangle)\, e^{-I_m - I_{\rm gh}}$$

  - $\mathcal{V}_1 \cdots \mathcal{V}_n$ are BRST-invariant vertex operators.
  - Picture Changing Operator formalism (Friedan, Martinec, Shenker 1986)
    - ⋆ may be obtained as singular limit for χ supported at points
    - ⋆ globally regular reformulation via "vertical integration" (Sen, Witten 2016)

Loop momenta and Chiral amplitudes

- h **independent loop momenta** p_I^μ **defined to flow across** $\mathfrak{A}_I$ **cycles**

$$p_I^\mu = \oint_{\mathfrak{A}_I} dz \partial_z x^\mu$$

- **Chiral Amplitudes** (ED, Phong 1988)
  - Massless NS bosons with factorized polarization tensor $\varepsilon_i^{\mu\tilde\mu} = \varepsilon_i^\mu \tilde\varepsilon_i^{\tilde\mu}$
  - Chiral amplitude at fixed loop momenta is given by

$$\mathcal{F}_R(\mathcal{J}, \varepsilon_i, k_i, p_I) = \left\langle \mathcal{V}_1 \cdots \mathcal{V}_N \, e^{p_I^\mu \oint_{\mathfrak{B}_I} dz \partial_z x^\mu} \, e^{\int_\Sigma H_{\tilde\theta}{}^z S_{z\theta}} \prod_A \delta(\langle B, H_A \rangle) d\mathfrak{m}^A \right\rangle$$

  - Correlation functions $\langle \cdots \rangle$ computed with chiral Green functions

- **Full Superstring Amplitudes**
  - obtained by pairing left and right and integrating over $\Gamma \in \mathfrak{M}_L \times \mathfrak{M}_R$

$$\mathcal{A}^{(h)}(\varepsilon_i, \tilde\varepsilon_i, k_i) = \int_{\mathbb{R}^{10}} dp_I^\mu \int_\Gamma \mathcal{F}_L(\tilde{\mathcal{J}}, \tilde\varepsilon_i, k_i, p_I^\mu) \, \mathcal{F}_R(\mathcal{J}, \varepsilon_i, k_i, p_I^\mu)$$

  - integration over vertex operator insertion points included in integration over Γ
  - cfr "double copy construction" in supergravity calculations

Parametrization of super moduli

- **Superconformal structure $\mathcal{J} \in \mathfrak{M}_h$ specified by transition functions**
  - Concrete calculations use parametrization by gravitino field $\chi_{\bar z}{}^\theta$
- **Local parametrization of moduli** (in conformal-invariant theory)
  - Conformal structure J with metric $g = |dz|^2$ in local coordinates $(z, \tilde z)$
  - deform conformal structure by Beltrami differential to $g' = |dz + \mu d\tilde z|^2$
  - realized in CFT by inserting $\int_\Sigma d\tilde z dz \, \mu_{\tilde z}{}^z \, T_{zz}$ to all orders in μ
- **Local parametrization of supermoduli** (in superconformal-invariant theory)
  - Start with $\Sigma_{\rm red}$ with complex structure given by $J \in \mathfrak{M}_{\rm red}$
  - Deform super conformal structure by inserting T and S

$$\int_{\Sigma_{\rm red}} d\tilde z dz \, \left(\mu_{\tilde z}{}^z T_{zz} + \chi_{\tilde z}{}^\theta S_{z\theta} \right)$$

  - χ and μ parametrized by local odd coordinates on $\mathfrak{M}_h$
- **For $h=2$, even spin structures, holó projection $\mathfrak{M}_2 \to \mathcal{M}_2$ exists**
  - via the super period matrix (ED, Phong 2001)
- **For $h \geq 5$ no holó projection $\mathfrak{M}_h \to \mathcal{M}_h$ exists** (Donagi, Witten 2013)

The super period matrix (even spin structures)

- **Start from conformal structure J for $\Sigma_{\rm red}$ with holó 1-forms ω_I**

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ} \qquad I,J=1,2$$

- **Deform to superconformal structure $\mathcal{J}$ on Σ with superholó forms $\hat\omega_I$**

$$\oint_{\mathfrak{A}_I} \hat\omega_J = \delta_{IJ} \qquad \oint_{\mathfrak{B}_I} \hat\omega_J = \hat\Omega_{IJ} \qquad I,J=1,2$$

  – Explicit formula for the super period matrix $\hat\Omega$ for even spin structure δ

$$\hat\Omega_{IJ} = \Omega_{IJ} - \frac{i}{8\pi}\int_{\Sigma^2_{\rm red}} \omega_I(z)\chi(z)S_\delta(z,w|\Omega)\chi(w)\omega_J(w) + \int_{\Sigma_{\rm red}} \mu\,\omega_I\omega_J$$

  – $\hat\Omega_{IJ}$ is locally supersymmetric; $\hat\Omega_{IJ}=\hat\Omega_{JI}$; and $\mathrm{Im}\,\hat\Omega>0$
  – Every $\hat\Omega$ corresponds to an ordinary Riemann surface
  – Szegö kernel $S_\delta(z,w|\Omega)$ is non-singular in the interior of $\mathcal{M}_2$

$\Rightarrow$ **Projection using $\hat\Omega$ is holomorphic and natural for genus 2**

Projecting and pairing Chiral Amplitudes

- **Chiral Amplitudes on $\mathfrak{M}_2$**
  - Natural parametrization of $\mathfrak{M}_2$ by $(\hat\Omega_{IJ}, \zeta^\alpha)$ (even spin structure δ)
  - involves measure $d\kappa[\delta](\hat\Omega,\zeta)$ and correlation functions $\mathcal{C}[\delta](\varepsilon_i,k_i,p_I|\hat\Omega,\zeta)$

- **Projection to chiral amplitudes on $\mathcal{M}_2$**
  - by integrating over ζ and summing over δ at fixed $\hat\Omega$

$$\mathcal{R}(\varepsilon_i,k_i,p_I|\hat\Omega) = \sum_\delta \int_\zeta d\kappa[\delta](\hat\Omega,\zeta)\,\mathcal{C}[\delta](\varepsilon_i,k_i,p_I|\hat\Omega,\zeta)$$

$$\mathcal{L}(\tilde\varepsilon_i,k_i,p_I|\hat\Omega) = \sum_{\tilde\delta} \int_{\tilde\zeta} d\kappa[\tilde\delta](\hat\Omega,\tilde\zeta)\,\mathcal{C}[\tilde\delta](\tilde\varepsilon_i,k_i,p_I|\hat\Omega,\tilde\zeta)$$

  - for heterotic, $\mathcal{L}$ is chiral half of bosonic string, has no integral in $\tilde\zeta$
  - phase factors determined by $Sp(4,\mathbb{Z})$ modular invariance

- **Pairing left and right chiral amplitudes, integrating over p_I and $\hat\Omega$**

$$\mathcal{A}^{(2)}(\varepsilon_i,\tilde\varepsilon_i,k_i) = \int_{\mathcal{M}_2} d\hat\Omega \int dp_I^\mu\, \mathcal{R}(\varepsilon_i,k_i,p_I|\hat\Omega)\,\overline{\mathcal{L}(\tilde\varepsilon_i,k_i,p_I|\hat\Omega)}$$

  - Integral over p_I is Gaussian and can be carried out explicitly.

Genus two

- **Siegel Upper half space $\mathcal{S}_2$**

$$\mathcal{S}_2 = \{\Omega_{IJ} = \Omega_{JI} \in \mathbb{C} \text{ with } I, J = 1, 2 \text{ and } Y = \mathrm{Im}\Omega > 0\}$$

  – $Sp(4, \mathbb{R})$ acts by $\Omega \to (A\Omega + B)(C\Omega + D)^{-1}$

$$M^t J M = J \qquad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

  – $\mathcal{S}_2$ has $Sp(4, \mathbb{R})$-invariant metric ds_2^2 and volume form $d\mu_2$

$$ds_2^2 = \sum_{I,J,K,L=1,2} Y^{-1}_{IJ}\, d\bar\Omega_{JK}\, Y^{-1}_{KL}\, d\Omega_{LI}$$

- **Compact Riemann surfaces Σ**
  – Choose canonical homology basis of $\mathfrak{A}_I, \mathfrak{B}_I$ cycles for $H_1(\Sigma, \mathbb{Z})$.
  – ω_I dual holomorphic (1,0) forms,

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ}$$

  – Riemann relations imply $\Omega \in \mathcal{S}_2$;
  – Modular group $Sp(4, \mathbb{Z})$; moduli space $\mathcal{M}_2 = \mathcal{S}_2 / Sp(4, \mathbb{Z})$.

Genus-two Type II four-graviton amplitude

- **Type II four-graviton amplitude** (ED, Phong 2001 – 2005)

$$\mathcal{A}^{(2)}(\varepsilon_i, \tilde\varepsilon_i, k_i) = g_s^2 \, \mathcal{K} \tilde{\mathcal{K}} \int_{\mathcal{M}_2} d\mu_2 \, \mathcal{B}^{(2)}(s_{ij}|\Omega)$$

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \bar{\mathcal{Y}}}{(\det \mathrm{Im}\,\Omega)^2} \exp \Big(\sum_{i<j} s_{ij} \, G(z_i, z_j|\Omega) \Big)$$

  – $G(z_i, z_j)$ is the genus-two scalar Green function;
  – $\Delta(z_i, z_j)$ is a bi-holomorphic form independent of s, t, u.

$$\Delta(z,w) = \omega_1(z) \wedge \omega_2(w) - \omega_2(z) \wedge \omega_1(w)$$

$$\begin{aligned} \mathcal{Y} = \; & (t-u)\Delta(z_1,z_2) \wedge \Delta(z_3,z_4) + (s-t)\Delta(z_1,z_3) \wedge \Delta(z_4,z_2) \\ & +(u-s)\Delta(z_1,z_4) \wedge \Delta(z_2,z_3) \end{aligned}$$

  – reproduced (with fermions) in pure spinor formulation (Berkovits, Mafra 2005)

- **Singularity structure**
  – For fixed Ω integrations over Σ produce poles in $\mathcal{B}$ at positive integers s_{ij}.
  – The integral over Ω requires analytic continuation beyond $\mathrm{Re}(s_{ij}) = 0$.
  – Branch cuts in s_{ij} starting at integers produced from $\Omega_{11}, \Omega_{22} \to i\infty$

Genus-two Heterotic four-graviton amplitude

- **Heterotic four NS boson amplitude at genus 2** (ED, Phong 2005)

$$\mathcal{A}^{(2)}_{\mathcal{O}}(\varepsilon_i, \tilde\varepsilon_i, k_i) = g_s^2 \, \mathcal{K} \int_{\mathcal{M}_2} d\mu_2 \, \mathcal{B}^{(2)}_{\mathcal{O}}(\tilde\varepsilon_i, k_i|\Omega)$$

$$\mathcal{B}^{(2)}_{\mathcal{O}}(\tilde\varepsilon_i, k_i|\Omega) = \int_{\Sigma^4} \frac{\mathcal{Y} \wedge \overline{\mathcal{W}_{\mathcal{O}}(\tilde\varepsilon_i, k_i)}}{(\det \mathrm{Im}\Omega)^2 \, \overline{\Psi_{10}(\Omega)}} \exp\Big(\sum_{i<j} s_{ij} \, G(z_i, z_j) \Big)$$

  – $\Psi_{10}(\Omega)$ is the Igusa cusp form.

- **Dependence of the operator $\mathcal{O}$ on the channel:**
  - ⋆ 4 gravitons $\mathcal{R}^4$
  - ⋆ 2 gravitons + 2 gauge bosons $\mathcal{R}^2 \mathrm{tr}(\mathcal{F}^2)$;
  - ⋆ 4 gauge bosons $(\mathrm{tr}\mathcal{F}^2)^2$
  - ⋆ 4 gauge bosons $\mathrm{tr}(\mathcal{F}^4)$

  – For example,

$$\mathcal{W}_{\mathcal{R}^4}(\tilde\varepsilon_i, k_i) = \frac{\langle \prod_{i=1}^4 \tilde\varepsilon_i \cdot \bar\partial \tilde x(z_i) \, e^{ik_i \cdot \tilde x(z_i)} \rangle}{\langle \prod_{i=1}^4 e^{ik_i \cdot \tilde x(z_i)} \rangle}$$

  – Gauge parts are obtained by the correlators of the current $(0,1)$-forms.

Singularities in the projection $\overline{\mathfrak{M}}_2 \to \overline{\mathcal{M}}_2$

- **Projection $\mathfrak{M}_2 \to \mathcal{M}_2$ is holó, but integration extends to boundary**
  - are there singularities in the projection $\overline{\mathfrak{M}}_2 \to \overline{\mathcal{M}}_2$?

$$\Omega = \begin{pmatrix} \tau & u \\ u & \sigma \end{pmatrix} \qquad \begin{array}{ll} u \to 0 & \text{separating node} \\ \sigma \to i\infty & \text{non-separating node} \end{array}$$

  - Key ingredient in $\hat{\Omega}$ is the Szegö kernel

$$S_\delta(z,w|\Omega) = \frac{\vartheta[\delta](z-w|\Omega)}{\vartheta[\delta](0|\Omega)E(z,w)}$$

  - As $u \to 0$ we have $\vartheta[\delta](0|\Omega) \to \vartheta[\delta_1](0|\tau)\,\vartheta[\delta_2](0|\tau)$
  - Even $\delta = [\delta_1, \delta_2]$ with δ_1, δ_2 odd produces a singularity in S_δ and $\hat{\Omega}$

- **Physical effects**
  - singularity killed by ψ-zero modes in $\mathbb{R}^{10}$ (space-time susy)
  - contribution when susy is broken by radiative corrections (Witten 2013)
  - Two-loop vacuum energy in Heterotic strings on CY orbifold $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$
    - ⋆ is zero for $E_8 \times E_8 \to E_6 \times E_8$ with unbroken susy
    - ⋆ non-zero for $\mathrm{Spin}(32)/\mathbb{Z}_2 \to SO(26) \times U(1)$ with broken susy

(Atick, Dixon, Sen 1988; Dine, Seiberg, Witten; ED, Phong 2013; Berkovits, Witten 2014)

Singularities in the projection $\mathfrak{M}_3 \to \mathcal{M}_3$

- **Some basic structure theorems**
  - A hyper-elliptic surface is a branched double cover of the sphere S^2;
  - All genus 1 and all genus 2 surfaces are hyper-elliptic;
  - Hyper-elliptic surfaces form a co-dim 1 sub-variety in the interior of $\mathcal{M}_3$
    (referred to as the hyper-elliptic divisor)

- **The genus-three period matrix (for even spin structure)**

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int\int \omega_I(z)\chi(z)S_\delta(z,w|\Omega)\chi(w)\omega_J(w) + \mathcal{O}(\chi^4)$$

  - For Ω on the hyper-elliptic divisor of $\mathfrak{M}_3$
    there always exists an even spin structure δ such that $\vartheta[\delta](0|\Omega) = 0$
  - the presence of the extra Dirac zero modes kills effects of this singularity

$\Rightarrow$ Beautiful proposal for the genus 3 superstring measure
(Cacciatori, Dalla Piazza, van Geemen 2008)

– Another even δ does produce a *subtle singularity* in $\hat{\Omega}$ (Witten 2015)