Stratifying On-Shell Cluster Varieties

Jacob L. Bourjaily

*Amplitudes* 2018 Summer School
QMAP, University of California, Davis
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Organization and Outline

1. The Amalgamation of On-Shell Diagrams
   - Basic Building Blocks: $S$-Matrices for Three Massless Particles

2. Building-Up the Grassmannian Correspondence: On-Shell Varieties
   - Grassmannian Representations of On-Shell Functions
   - Iterative Construction of Grassmannian ‘On-Shell’ Varieties
   - Characteristics of Grassmannian Representations

3. The Classification of On-Shell (Cluster) Varieties
   - Warm-Up: Classifying On-Shell Functions of $G(2,n)$
   - Definitions, Stratifications, and Conjectures
   - Application: the Stratification of On-Shell Varieties in $G(3,6)$

4. Conclusions and Future Directions
Amalgamating Diagrams from Three-Particle Amplitudes

Recall that on-shell diagrams built out of three-point amplitudes are always meaningful functions—even when the result is non-planar
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\[
\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle
\]
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\[
\left( \langle 91 \rangle \langle 23 \rangle \langle 46 \rangle - \langle 16 \rangle \langle 34 \rangle \langle 29 \rangle \right)^2 \frac{\delta^{2 \times 4} (\lambda \cdot \tilde{\eta}) \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda})}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle}
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\[
\begin{align*}
\langle 91 \rangle \langle 23 \rangle \langle 46 \rangle - \langle 16 \rangle \langle 34 \rangle \langle 29 \rangle & \quad \delta^{2 \times 4} (\lambda \cdot \tilde{\eta}) \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \\
\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle
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\[
\begin{align*}
\langle 91 \rangle & \langle 23 \rangle \langle 46 \rangle - \langle 16 \rangle \langle 34 \rangle \langle 29 \rangle \\
\langle 12 \rangle & \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle \\
= & \frac{\delta^{2 \times 4} (\lambda \cdot \tilde{\eta}) \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda})}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle}
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\[
\frac{\langle 91 \rangle \langle 23 \rangle \langle 46 \rangle - \langle 16 \rangle \langle 34 \rangle \langle 29 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle} \left( \delta^{2 \times 4} (\lambda \cdot \tilde{\eta}) \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \right)
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance uniquely fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
\langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1} \\
\propto h_1 + h_2 + h_3 \leq 0
\end{align*}
\]

\[
\begin{align*}
[12]^{h_1+h_2-h_3} \langle 23 \rangle^{h_2+h_3-h_1} \langle 31 \rangle^{h_3+h_1-h_2} \\
\propto h_1 + h_2 + h_3 \geq 0
\end{align*}
\]
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\[
\begin{align*}
\text{Diagram 1:} & \quad \frac{\langle 2\ 3 \rangle^4}{\langle 1\ 2\rangle \langle 2\ 3\rangle \langle 3\ 1\rangle} \delta^{2\times2}(\lambda \cdot \tilde{\lambda}) \\
\text{Diagram 2:} & \quad \frac{[2\ 3]^4}{[1\ 2][2\ 3][3\ 1]} \delta^{2\times2}(\lambda \cdot \tilde{\lambda})
\end{align*}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical
dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
1 & \rightarrow 2 &= \frac{\langle 2 \ 3 \rangle^4}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \langle 3 \ 1 \rangle} \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv A_3 (+, -, -) \\
1 & \rightarrow 3 \rightarrow 2 &= \frac{\langle 2 \ 3 \rangle^4}{[1 \ 2] [2 \ 3] [3 \ 1]} \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv A_3 (-, +, +)
\end{align*}
\]
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1 & \rightarrow 2 \quad = \quad \frac{\langle 2 \ 3 \rangle^4}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \langle 3 \ 1 \rangle} \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv A_3 (+, -, -) \\
1 & \rightarrow 3 \quad 2 \quad = \quad \frac{[2 \ 3]^4}{[1 \ 2] [2 \ 3] [3 \ 1]} \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv A_3 (-, +, +)
\end{align*}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance \textbf{uniquely} fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
1 & \quad \quad 2 \\
\quad \quad 3 & \quad \quad 2
\end{align*}
\]

\[
1 = \frac{\langle 3 1 \rangle \langle 2 3 \rangle^3}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 1 \rangle} \quad \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv A_3 (+ \frac{1}{2}, - \frac{1}{2}, -)
\]

\[
1 \quad \quad 3 \\
\quad \quad 3 \quad \quad 2
\]

\[
1 = \frac{[3 1][2 3]^3}{[1 2][2 3][3 1]} \quad \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv A_3 (- \frac{1}{2}, + \frac{1}{2}, +)
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Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
1 & \rightarrow 2 & = & \frac{\langle 3 \ 1 \rangle \langle 2 \ 3 \rangle^3}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \langle 3 \ 1 \rangle} \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv A_3 \left( +\frac{1}{2}, -\frac{1}{2}, - \right) \\
1 & \rightarrow 3 & = & \frac{[3 \ 1][2 \ 3]^3}{[1 \ 2][2 \ 3][3 \ 1]} \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv A_3 \left( -\frac{1}{2}, +\frac{1}{2}, + \right)
\end{align*}
\]

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Part III: Stratifying On-Shell Cluster Varieties
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
1 & \rightarrow 2 \\
3 & \rightarrow 1
\end{align*}
\]

\[
\begin{align*}
\delta^{2\times4} (\lambda \cdot \tilde{\eta}) & \equiv A^{(2)}_3 \\
\frac{\delta^{2\times4} (\lambda \cdot \tilde{\eta})}{\langle 1 2\rangle \langle 2 3\rangle \langle 3 1\rangle} & = \frac{\delta^{1\times4} (\tilde{\lambda} \cdot \tilde{\eta})}{[1 2] [2 3] [3 1]} \equiv A^{(1)}_3
\end{align*}
\]

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Momentum conservation and Poincaré-invariance uniquely fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
\text{Diagram 1} & : \quad 1 \quad = \quad \frac{\delta^{2\times4}(\lambda \cdot \eta)}{\langle 1 \; 2 \rangle \langle 2 \; 3 \rangle \langle 3 \; 1 \rangle} \quad \delta^{2\times2}(\lambda \cdot \tilde{\lambda}) \equiv A^{(2)}_3 \\
\text{Diagram 2} & : \quad 1 \quad = \quad \frac{\delta^{1\times4}(\tilde{\lambda}^\perp \cdot \eta)}{[1 \; 2] \; [2 \; 3] \; [3 \; 1]} \quad \delta^{2\times2}(\lambda \cdot \tilde{\lambda}) \equiv A^{(1)}_3
\end{align*}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance uniquely fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
\text{Tree 1:} & \quad \delta^{2 \times 4} \left( \lambda \cdot \tilde{\eta} \right) \\
& \quad \frac{\delta^{2 \times 4} \left( \lambda \cdot \tilde{\eta} \right)}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 1 \rangle} \delta^{2 \times 2} \left( \lambda \cdot \tilde{\lambda} \right) \equiv A_3^{(2)} \\
\text{Tree 2:} & \quad \delta^{1 \times 4} \left( \tilde{\lambda}^\perp \cdot \tilde{\eta} \right) \\
& \quad \frac{\delta^{1 \times 4} \left( \tilde{\lambda}^\perp \cdot \tilde{\eta} \right)}{[1 2] [2 3] [3 1]} \delta^{2 \times 2} \left( \lambda \cdot \tilde{\lambda} \right) \equiv A_3^{(1)}
\end{align*}
\]
Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex.
Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex, (and to specify *which* of the solutions to three-particle kinematics to use)
Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex, (and to specify *which* of the solutions to three-particle kinematics to use) we introduce **auxiliary** $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:
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$$A_3^{(2)} = \frac{\delta^{2\times4} (\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2\times2} (\lambda \cdot \tilde{\lambda})$$
In order to linearize momentum conservation at each three-particle vertex, (and to specify which of the solutions to three-particle kinematics to use) we introduce auxiliary $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

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$$B \equiv \begin{pmatrix} b_1^1 & b_2^1 & b_3^1 \\ b_1^2 & b_2^2 & b_3^2 \\ b_1^3 & b_2^3 & b_3^3 \end{pmatrix}$$

$$\mathcal{A}^{(2)}_3 = \frac{\delta^{2 \times 4} (\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda})$$
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In order to linearize momentum conservation at each three-particle vertex, (and to specify which of the solutions to three-particle kinematics to use) we introduce auxiliary $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

$$
\begin{align*}
A^{(2)}_3 &= \frac{\delta^{2 \times 4} (\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \\
&= \int \frac{d^2 B}{\text{vol}(G L_2)} \frac{\delta^{2 \times 4} (B \cdot \tilde{\eta})}{(12) (23) (31)} \delta^{2 \times 2} (B \cdot \tilde{\lambda}) \delta^{1 \times 2} (\lambda \cdot B^\perp)
\end{align*}
$$
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In order to linearize momentum conservation at each three-particle vertex, (and to specify which of the solutions to three-particle kinematics to use) we introduce auxiliary $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

$$B \equiv \begin{pmatrix} b_1^1 & b_2^1 & b_3^1 \\ b_1^2 & b_2^2 & b_3^2 \\ b_1^3 & b_2^3 & b_3^3 \end{pmatrix}$$

$$W \equiv \begin{pmatrix} w_1^1 & w_2^1 & w_3^1 \end{pmatrix}$$

The amplitude $\mathcal{A}_3^{(2)}$ is given by:

$$\mathcal{A}_3^{(2)} = \frac{\delta^{2\times4}(\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2\times2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^{2\times3}B}{\text{vol}(GL_2)} \frac{\delta^{2\times4}(B \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2\times2}(B \cdot \tilde{\lambda}) \delta^{1\times2}(\lambda \cdot B^\perp)$$

The amplitude $\mathcal{A}_3^{(1)}$ is given by:

$$\mathcal{A}_3^{(1)} = \frac{\delta^{1\times4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12][23][31]} \delta^{2\times2}(\lambda \cdot \tilde{\lambda})$$
Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex, (and to specify *which* of the solutions to three-particle kinematics to use) we introduce **auxiliary** $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

$$B = \begin{pmatrix} b_1^1 & b_1^2 & b_1^3 \\ b_2^1 & b_2^2 & b_2^3 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} w_1^1 \\ w_1^2 \\ w_1^3 \end{pmatrix}$$

$$A_3^{(2)} = \frac{\delta^{2\times4}(\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2\times2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^2\times3B}{\text{vol}(GL_2)} \frac{\delta^{2\times4}(B \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2\times2}(B \cdot \tilde{\lambda}) \delta^{1\times2}(\lambda \cdot B^\perp)$$

$$A_3^{(1)} = \frac{\delta^{1\times4}(\tilde{\lambda} \cdot \tilde{\eta})}{[12][23][31]} \delta^{2\times2}(\lambda \cdot \tilde{\lambda})$$
Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex, (and to specify which of the solutions to three-particle kinematics to use) we introduce **auxiliary** $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

$$B \equiv \begin{pmatrix} b_1^1 & b_2^1 & b_3^1 \\ b_1^2 & b_2^2 & b_3^2 \\ b_1^3 & b_2^3 & b_3^3 \end{pmatrix}$$

$$W \equiv \begin{pmatrix} w_1^1 \\ w_2^1 \\ w_3^1 \end{pmatrix}$$

$$A^{(2)}_3 = \frac{\delta^{2 \times 4}(\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^2 B}{\text{vol}(GL_2)} \frac{\delta^{2 \times 4}(B \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2}(B \cdot \tilde{\lambda}) \delta^{1 \times 2}(\lambda \cdot B^\perp)$$

$$A^{(1)}_3 = \frac{\delta^{1 \times 4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12][23][31]} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda})$$
Grassmannian Representations of Three-Point Amplitudes

In order to linearize momentum conservation at each three-particle vertex, (and to specify which of the solutions to three-particle kinematics to use) we introduce auxiliary \( B \in G(2,3) \) and \( W \in G(1,3) \) for each vertex:

\[
\begin{align*}
1 \quad &\leftrightarrow B = \begin{pmatrix} b_1^1 & b_1^2 & b_1^3 \\ b_2^1 & b_2^2 & b_2^3 \end{pmatrix} \\
1 \quad &\leftrightarrow W = \begin{pmatrix} w_1^1 & w_1^2 & w_1^3 \end{pmatrix}
\end{align*}
\]

\[
\mathcal{A}^{(2)}_3 = \frac{\delta^{2 \times 4} \left( \lambda \cdot \tilde{\eta} \right)}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2} \left( \lambda \cdot \tilde{\lambda} \right) \equiv \int \frac{d^2}{\text{vol}(GL_2)} \delta^{2 \times 4} \left( B \cdot \tilde{\eta} \right) \delta^{2 \times 2} \left( B \cdot \tilde{\lambda} \right) \delta^{1 \times 2} \left( \lambda \cdot B^\perp \right)
\]

\[
\mathcal{A}^{(1)}_3 = \frac{\delta^{1 \times 4} \left( \tilde{\lambda}^\perp \cdot \tilde{\eta} \right)}{[12][23][31]} \delta^{2 \times 2} \left( \lambda \cdot \tilde{\lambda} \right) \equiv \int \frac{d^1}{\text{vol}(GL_1)} \delta^{1 \times 4} \left( W \cdot \tilde{\eta} \right) \delta^{1 \times 2} \left( W \cdot \tilde{\lambda} \right) \delta^{2 \times 2} \left( \lambda \cdot W^\perp \right)
\]
Grassmannian Representations of Three-Point Amplitudes

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\[
1 \rightarrow B \equiv \begin{pmatrix} b_1^1 & b_2^1 & b_3^1 \\ b_1^2 & b_2^2 & b_3^2 \end{pmatrix} \\
2 \rightarrow W \equiv \begin{pmatrix} w_1^1 & w_2^1 & w_3^1 \end{pmatrix}
\]

\[
\mathcal{A}_3^{(2)} = \frac{\delta^{2 \times 4}(\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^2 \times 3 B}{\text{vol}(GL_2)} \frac{\delta^{2 \times 4}(B \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2}(B \cdot \tilde{\lambda}) \delta^{1 \times 2}(\lambda \cdot B^\perp)
\]

\[
\mathcal{A}_3^{(1)} = \frac{\delta^{1 \times 4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12][23][31]} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^1 \times 3 W}{\text{vol}(GL_1)} \frac{\delta^{1 \times 4}(W \cdot \tilde{\eta})}{(1) (2) (3)} \delta^{1 \times 2}(W \cdot \tilde{\lambda}) \delta^{2 \times 2}(\lambda \cdot W^\perp)
\]
Grassmannian Representations of Three-Point Amplitudes

In order to linearize momentum conservation at each three-particle vertex, (and to specify which of the solutions to three-particle kinematics to use) we introduce auxiliary $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

\[ B \equiv \begin{pmatrix} b_1^1 & b_2^1 & b_3^1 \\ b_1^2 & b_2^2 & b_3^2 \end{pmatrix} \]
\[ W \equiv \begin{pmatrix} w_1^1 & w_1^1 & w_1^1 \end{pmatrix} \]

\[ \mathcal{A}_3^{(2)} = \frac{\delta^{2\times 4}(\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2\times 2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^2 \times 3 B}{\text{vol}(GL_2)} \frac{\delta^{2\times 4}(B \cdot \tilde{\eta})}{(12)(23)(31)} \delta^{2\times 2}(B \cdot \tilde{\lambda}) \delta^{1\times 2}(\lambda \cdot B^\perp) \]

\[ \mathcal{A}_3^{(1)} = \frac{\delta^{1\times 4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12][23][31]} \delta^{2\times 2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^1 \times 3 W}{\text{vol}(GL_1)} \frac{\delta^{1\times 4}(W \cdot \tilde{\eta})}{(1)(2)(3)} \delta^{1\times 2}(W \cdot \tilde{\lambda}) \delta^{2\times 2}(\lambda \cdot W^\perp) \]
Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex, (and to specify *which* of the solutions to three-particle kinematics to use) we introduce **auxiliary** $B \in G(2,3)$ and $W \in G(1,3)$ for each vertex:

\[
\begin{align*}
1 \quad & \quad \iff \quad B = \begin{pmatrix} b_1^1 & b_2^1 & b_3^1 \\ b_1^2 & b_2^2 & b_3^2 \end{pmatrix} \\
3 \quad & \quad \iff \quad W = \begin{pmatrix} w_1^1 & w_2^1 & w_3^1 \end{pmatrix}
\end{align*}
\]

\[
\mathcal{A}_3^{(2)} = \frac{\delta^{2\times4}(\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2\times2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^2 \times 3 B}{\text{vol}(GL_2)} \frac{\delta^{2\times4}(B \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2\times2}(B \cdot \tilde{\lambda}) \delta^{1\times2}(\lambda \cdot B^\perp)
\]

\[
\mathcal{A}_3^{(1)} = \frac{\delta^{1\times4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12][23][31]} \delta^{2\times2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^1 \times 3 W}{\text{vol}(GL_1)} \frac{\delta^{1\times4}(W \cdot \tilde{\eta})}{(1)(2)(3)} \delta^{2\times2}(W \cdot \tilde{\lambda}) \delta^{1\times2}(\lambda \cdot W^\perp)
\]
Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex, (and to specify *which* of the solutions to three-particle kinematics to use) we introduce **auxiliary** $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

\[
\begin{align*}
\text{1} & \quad \leftrightarrow 
\begin{pmatrix}
1 & b_1^1 & b_1^2 & b_1^3 \\
b_2^1 & 2 & b_2^2 & b_2^3 \\
b_3^1 & b_3^2 & 3
\end{pmatrix} \\
\text{2} & \quad \leftrightarrow 
\begin{pmatrix}
w_1^1 & 1 & w_1^2 \\
w_2^1 & w_2^2 & w_2^3 \\
w_3^1 & 3 & w_3^3
\end{pmatrix} \\
\text{3} & \quad \leftrightarrow 
\begin{pmatrix}
\lambda \cdot \bar{\eta} \\
\lambda \cdot B_\perp
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\mathcal{A}_3^{(2)} &= \frac{\delta^{2\times4}(\lambda \cdot \bar{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2\times2}(\lambda \cdot \bar{\lambda}) \\
&= \int \frac{d^2 B}{\text{vol}(GL_2)} \frac{\delta^{2\times4}(B \cdot \bar{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2\times2}(B \cdot \bar{\lambda}) \delta^{1\times2}(\lambda \cdot B_\perp)
\end{align*}
\]

\[
\begin{align*}
\mathcal{A}_3^{(1)} &= \frac{\delta^{1\times4}(\bar{\lambda}^\perp \cdot \bar{\eta})}{[12][23][31]} \delta^{2\times2}(\lambda \cdot \bar{\lambda}) \\
&= \int \frac{d^3 W}{\text{vol}(GL_1)} \frac{\delta^{1\times4}(W \cdot \bar{\eta})}{(1)(2)(3)} \delta^{2\times2}(W \cdot \bar{\lambda}) \delta^{1\times2}(\lambda \cdot W_\perp)
\end{align*}
\]
Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex, (and to specify *which* of the solutions to three-particle kinematics to use) we introduce auxiliary $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

$$
\begin{align*}
1 & \quad \leftrightarrow \quad B = \begin{pmatrix} b_1^1 & b_2^1 & b_3^1 \\ b_1^2 & b_2^2 & b_3^2 \end{pmatrix} \quad & & \begin{pmatrix} w_1^1 & w_1^1 & w_3^1 \\ w_2^1 & w_2^1 & w_3^2 \end{pmatrix} \\
2 & & & 3
\end{align*}
$$

$$
\mathcal{A}_3^{(2)} = \frac{\delta^{2 \times 4} (\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv \int_\text{vol(GL}_2) \frac{d^2 \times 3 B}{(12)(23)(31)} \frac{\delta^{2 \times 4} (B \cdot \tilde{\eta})}{\delta^{2 \times 2} (B \cdot \tilde{\lambda}) \delta^{1 \times 2} (\lambda \cdot B^\perp)}
$$

$$
\mathcal{A}_3^{(1)} = \frac{\delta^{1 \times 4} (\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12][23][31]} \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv \int_\text{vol(GL}_1) \frac{d^1 \times 3 W}{(1)(2)(3)} \frac{\delta^{1 \times 4} (W \cdot \tilde{\eta})}{\delta^{1 \times 2} (W \cdot \tilde{\lambda}) \delta^{2 \times 2} (\lambda \cdot W^\perp)}
$$
Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex, (and to specify *which* of the solutions to three-particle kinematics to use) we introduce **auxiliary** $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

$$ B \equiv \begin{pmatrix} 1 & 0 & b_3^1 \\ 0 & 1 & b_3^2 \\ b_3^1 & b_3^2 & 0 \end{pmatrix} $$

$$ W \equiv \begin{pmatrix} 1 & w_2^1 & w_3^1 \\ w_2^1 & w_3^1 & 0 \end{pmatrix} $$

\[
\mathcal{A}^{(2)}_3 = \frac{\delta^{2 \times 4}(\lambda \cdot \tilde{\eta})}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d b_3^1}{b_3^1} \wedge \frac{d b_3^2}{b_3^2} \delta^{2 \times 4}(B \cdot \tilde{\eta}) \delta^{2 \times 2}(B \cdot \tilde{\lambda}) \delta^{1 \times 2}(\lambda \cdot B^\perp) \\
\mathcal{A}^{(1)}_3 = \frac{\delta^{1 \times 4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12][23][31]} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d w_2^1}{w_2^1} \wedge \frac{d w_3^1}{w_3^1} \delta^{1 \times 4}(W \cdot \tilde{\eta}) \delta^{1 \times 2}(W \cdot \tilde{\lambda}) \delta^{2 \times 2}(\lambda \cdot W^\perp)
\]
Grassmannian Representations of Three-Point Amplitudes

In order to linearize momentum conservation at each three-particle vertex, (and to specify which of the solutions to three-particle kinematics to use) we introduce auxiliary $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

$$
B \equiv \begin{pmatrix} \frac{1}{b_1^1} & 1 & 0 \\ \frac{1}{b_1^2} & 0 & 1 \end{pmatrix} \\
W \equiv \begin{pmatrix} \frac{1}{w_1^1} & 1 & \frac{1}{w_3^1} \end{pmatrix}
$$

$$
\mathcal{A}_3^{(2)} = \frac{\delta^{2 \times 4} (\lambda \cdot \bar{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2} (\lambda \cdot \bar{\lambda}) \equiv \int \frac{d b_1^1}{b_1^1} \wedge \frac{d b_1^2}{b_1^2} \delta^{2 \times 4} (B \cdot \bar{\eta}) \delta^{2 \times 2} (B \cdot \bar{\lambda}) \delta^{1 \times 2} (\lambda \cdot B^\perp)
$$

$$
\mathcal{A}_3^{(1)} = \frac{\delta^{1 \times 4} (\bar{\lambda} \cdot \bar{\eta})}{[12] [23] [31]} \delta^{2 \times 2} (\lambda \cdot \bar{\lambda}) \equiv \int \frac{d w_3^1}{w_3^1} \wedge \frac{d w_1^1}{w_1^1} \delta^{1 \times 4} (W \cdot \bar{\eta}) \delta^{1 \times 2} (W \cdot \bar{\lambda}) \delta^{2 \times 2} (\lambda \cdot W^\perp)
$$
Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex, (and to specify *which* of the solutions to three-particle kinematics to use) we introduce **auxiliary** $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

\[
\begin{align*}
1 & \leftrightarrow B = \begin{pmatrix} 0 & b_2^1 & 1 \\ 1 & b_2^2 & 0 \end{pmatrix} \\
1 & \leftrightarrow W = \begin{pmatrix} w_1^1 & w_2^1 & 1 \end{pmatrix}
\end{align*}
\]

\[
\mathcal{A}_3^{(2)} = \frac{\delta^{2\times4}(λ \cdot \tilde{η})}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \delta^{2\times2}(λ \cdot \tilde{λ}) = \int \frac{dB_2^1}{b_2^1} \wedge \frac{dB_2^2}{b_2^2} \delta^{2\times4}(B \cdot \tilde{η}) \delta^{2\times2}(B \cdot \tilde{λ}) \delta^{1\times2}(λ \cdot B^\perp)
\]

\[
\mathcal{A}_3^{(1)} = \frac{\delta^{1\times4}(\tilde{λ}^\perp \cdot \tilde{η})}{[12][23][31]} \delta^{2\times2}(λ \cdot \tilde{λ}) = \int \frac{dW_1^1}{w_1^1} \wedge \frac{dW_2^1}{w_2^1} \delta^{1\times4}(W \cdot \tilde{η}) \delta^{1\times2}(W \cdot \tilde{λ}) \delta^{2\times2}(λ \cdot W^\perp)
\]
Grassmannian Representations of Three-Point Amplitudes

In order to linearize momentum conservation at each three-particle vertex, (and to specify which of the solutions to three-particle kinematics to use) we introduce auxiliary $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

\[
\begin{align*}
A_3^{(2)} &= \frac{\delta^{2\times4}(\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2\times2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^{2\times3}B}{\text{vol}(GL_2)} \frac{\delta^{2\times4}(B \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2\times2}(B \cdot \tilde{\lambda}) \delta^{1\times2}(\lambda \cdot B^\perp) \\
A_3^{(1)} &= \frac{\delta^{1\times4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12][23][31]} \delta^{2\times2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^{1\times3}W}{\text{vol}(GL_1)} \frac{\delta^{1\times4}(W \cdot \tilde{\eta})}{(1)\langle 2 \rangle \langle 3 \rangle} \delta^{2\times2}(W \cdot \tilde{\lambda}) \delta^{1\times2}(\lambda \cdot W^\perp)
\end{align*}
\]
Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex, (and to specify which of the solutions to three-particle kinematics to use) we introduce **auxiliary** $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

$$B = \begin{pmatrix} b^1_1 & b^1_2 & b^1_3 \\ b^2_1 & b^2_2 & b^2_3 \end{pmatrix} \quad W = \begin{pmatrix} w^1_1 \\ w^1_2 \\ w^1_3 \end{pmatrix}$$

$$A_3^{(2)} = \frac{\delta^{2 \times 4}(\lambda \cdot \tilde{\eta})}{\langle 12\rangle \langle 23\rangle \langle 31\rangle} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^2 B}{\text{vol}(GL_2)} \frac{\delta^{2 \times 4}(B \cdot \tilde{\eta})}{\langle 12\rangle \langle 23\rangle \langle 31\rangle} \delta^{2 \times 2}(B \cdot \tilde{\lambda}) \delta^{1 \times 2}(\lambda \cdot B^\perp)$$

$$A_3^{(1)} = \frac{\delta^{1 \times 4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12][23][31]} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^1 W}{\text{vol}(GL_1)} \frac{\delta^{1 \times 4}(W \cdot \tilde{\eta})}{\langle 1\rangle \langle 2\rangle \langle 3\rangle} \delta^{1 \times 2}(W \cdot \tilde{\lambda}) \delta^{2 \times 2}(\lambda \cdot W^\perp)$$
Grassmannian Representations of Three-Point Amplitudes

In order to \textit{linearize} momentum conservation at each three-particle vertex, (and to specify \textit{which} of the solutions to three-particle kinematics to use) we introduce \textbf{auxiliary} $B \in G(2,3)$ and $W \in G(1,3)$ for each vertex:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
1 \quad \blacksquare \\
\end{array} \\
\begin{array}{c}
2 \\
\end{array} \\
\begin{array}{c}
3 \\
\end{array}
\end{array}
\end{align*}
\iff
\begin{align*}
\begin{array}{c}
1 \quad \blacksquare \\
\begin{array}{c}
2 \\
\end{array} \\
\begin{array}{c}
3 \\
\end{array}
\end{array}
\end{align*}

\[
\begin{align*}
\mathcal{A}_3^{(2)} &= \frac{\delta^{2\times4}(\lambda \cdot \tilde{\eta})}{\langle12\rangle\langle23\rangle\langle31\rangle} \delta^{2\times2}(\lambda \cdot \tilde{\lambda}) = \int \frac{d^2\times3 B}{\text{vol}(GL_2)} \frac{\delta^{2\times4}(B \cdot \tilde{\eta})}{\langle12\rangle\langle23\rangle\langle31\rangle} \delta^{2\times2}(B \cdot \tilde{\lambda}) \delta^{1\times2}(\lambda \cdot B^\perp)
\end{align*}
\]

\[
\begin{align*}
\mathcal{A}_3^{(1)} &= \frac{\delta^{1\times4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12][23][31]} \delta^{2\times2}(\lambda \cdot \tilde{\lambda}) = \int \frac{d^1\times3 W}{\text{vol}(GL_1)} \frac{\delta^{1\times4}(W \cdot \tilde{\eta})}{(1)(2)(3)} \delta^{1\times2}(W \cdot \tilde{\lambda}) \delta^{2\times2}(\lambda \cdot W^\perp)
\end{align*}
\]
Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex, (and to specify which of the solutions to three-particle kinematics to use) we introduce auxiliary $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

$$B \equiv \begin{pmatrix} b_1^1 & b_2^1 & b_3^1 \\ b_1^2 & b_2^2 & b_3^2 \end{pmatrix}$$

$$W \equiv \begin{pmatrix} w_1^1 \\ w_2^1 \\ w_3^1 \end{pmatrix}$$

$$A_3^{(2)} = \frac{\delta^{2 \times 4}(\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^2 \times 3 B}{\text{vol}(GL_2)} \frac{\delta^{2 \times 4}(B \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2}(B \cdot \tilde{\lambda}) \delta^{1 \times 2}(\lambda \cdot B^\perp)$$

$$A_3^{(1)} = \frac{\delta^{1 \times 4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12][23][31]} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^1 \times 3 W}{\text{vol}(GL_1)} \frac{\delta^{1 \times 4}(W \cdot \tilde{\eta})}{(1) (2) (3)} \delta^{2 \times 2}(W \cdot \tilde{\lambda}) \delta^{1 \times 2}(\lambda \cdot W^\perp)$$
Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex, (and to specify *which* of the solutions to three-particle kinematics to use) we introduce **auxiliary** $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

$$
\begin{align*}
\text{1} & \quad \leftrightarrow \quad B = \begin{pmatrix}
    b_1^1 & b_2^1 & b_3^1 \\
    b_1^2 & b_2^2 & b_3^2
\end{pmatrix} \\
\text{1} & \quad \leftrightarrow \quad W = \begin{pmatrix}
    w_1^1 & w_2^1 & w_3^1
\end{pmatrix}
\end{align*}
$$

$$
A_3^{(2)} = \frac{\delta^{2 \times 4}(\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \\ = \int \frac{d^2 \times 3 B}{\text{vol}(GL_2)} \frac{\delta^{2 \times 4}(B \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2}(B \cdot \tilde{\lambda}) \delta^{1 \times 2}(\lambda \cdot B^\perp) \\ B \mapsto B^*
$$

$$
A_3^{(1)} = \frac{\delta^{1 \times 4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12][23][31]} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \\ = \int \frac{d^1 \times 3 W}{\text{vol}(GL_1)} \frac{\delta^{1 \times 4}(W \cdot \tilde{\eta})}{(1)(2)(3)} \delta^{1 \times 2}(W \cdot \tilde{\lambda}) \delta^{2 \times 2}(\lambda \cdot W^\perp)
$$
Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex, (and to specify *which* of the solutions to three-particle kinematics to use) we introduce **auxiliary** $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

$$
1 \rightarrow 2 \leftrightarrow B \equiv \begin{pmatrix} b_1^1 & b_2^1 & b_3^1 \\ b_1^2 & b_2^2 & b_3^2 \\ b_1^3 \end{pmatrix} 
$$

$$
1 \rightarrow 3 \leftrightarrow W \equiv \begin{pmatrix} w_1^1 & w_2^1 & w_3^1 \end{pmatrix}
$$

$$
\mathcal{A}_3^{(2)} = \frac{\delta^{2 \times 4}(\lambda \cdot \tilde{\eta})}{\langle 12 \langle 23 \langle 31 \rangle} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^2 \times B}{\text{vol}(GL_2)} \frac{\delta^{2 \times 4}(B \cdot \tilde{\eta})}{\langle 12 \langle 23 \langle 31 \rangle} \delta^{2 \times 2}(B \cdot \tilde{\lambda}) \delta^{1 \times 2}(\lambda \cdot B^\perp)
$$

$$
\mathcal{A}_3^{(1)} = \frac{\delta^{1 \times 4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12] [23] [31]} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^1 \times W}{\text{vol}(GL_1)} \frac{\delta^{1 \times 4}(W \cdot \tilde{\eta})}{\langle 1 \langle 2 \langle 3 \rangle} \delta^{1 \times 2}(W \cdot \tilde{\lambda}) \delta^{2 \times 2}(\lambda \cdot W^\perp)
$$
Grassmannian Representations of Three-Point Amplitudes

In order to linearize momentum conservation at each three-particle vertex, (and to specify which of the solutions to three-particle kinematics to use) we introduce auxiliary $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

$$
B \equiv \begin{pmatrix} b_1^1 & b_2^1 & b_3^1 \\ b_1^2 & b_2^2 & b_3^2 \\ b_1^3 & b_2^3 & b_3^3 \end{pmatrix}
$$

$$
W \equiv \begin{pmatrix} w_1^1 & w_2^1 & w_3^1 \end{pmatrix}
$$

\[ \mathcal{A}_3^{(2)} = \frac{\delta^{2\times 4}(\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2\times 2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^2 \times 3 B}{\text{vol}(GL_2)} \frac{\delta^{2\times 4}(B \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2\times 2}(B \cdot \tilde{\lambda}) \delta^{1\times 2}(\lambda \cdot B^\perp) \]

\[ \mathcal{A}_3^{(1)} = \frac{\delta^{1\times 4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12][23][31]} \delta^{2\times 2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^1 \times 3 W}{\text{vol}(GL_1)} \frac{\delta^{1\times 4}(W \cdot \tilde{\eta})}{\langle 1 \rangle \langle 2 \rangle \langle 3 \rangle} \delta^{1\times 2}(W \cdot \tilde{\lambda}) \delta^{2\times 2}(\lambda \cdot W^\perp) \]
Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex, (and to specify *which* of the solutions to three-particle kinematics to use) we introduce **auxiliary** $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

\[
\begin{align*}
1 & \quad \leftrightarrow \quad B = \left( b_1^1 \ b_2^1 \ b_3^1 \ b_1^2 \ b_2^2 \ b_3^2 \right) \\
2 & \quad \leftrightarrow \quad W = \left( w_1^1 \ w_2^1 \ w_3^1 \right)
\end{align*}
\]

\[
\mathcal{A}_3^{(2)} = \frac{\delta^{2 \times 4} (\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^2 \times 3 B}{\text{vol}(GL_2)} \frac{\delta^{2 \times 4} (B \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2} (B \cdot \tilde{\lambda}) \delta^{1 \times 2} (\lambda \cdot B^\perp)
\]

\[
\mathcal{A}_3^{(1)} = \frac{\delta^{1 \times 4} (\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12] [23] [31]} \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^1 \times 3 W}{\text{vol}(GL_1)} \frac{\delta^{1 \times 4} (W \cdot \tilde{\eta})}{(1) (2) (3)} \delta^{1 \times 2} (W \cdot \tilde{\lambda}) \delta^{2 \times 2} (\lambda \cdot W^\perp)
\]
Grassmannian Representations of Three-Point Amplitudes

In order to **linearize** momentum conservation at each three-particle vertex, (and to specify *which* of the solutions to three-particle kinematics to use) we introduce auxiliary $B \in G(2, 3)$ and $W \in G(1, 3)$ for each vertex:

$$1 \rightarrow B \equiv \begin{pmatrix} b_1^1 & b_1^2 & b_1^3 \\ b_2^1 & b_2^2 & b_2^3 \end{pmatrix}$$

$$1 \rightarrow W \equiv \begin{pmatrix} w_1^1 & w_2^1 & w_3^1 \end{pmatrix}$$

$$\mathcal{A}_3^{(2)} = \frac{\delta^{2 \times 4}(\lambda \cdot \bar{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2}(\lambda \cdot \bar{\lambda}) \equiv \int \frac{d^2 \times 3 B}{\text{vol}(GL_2)} \frac{\delta^{2 \times 4}(B \cdot \bar{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2}(B \cdot \bar{\lambda}) \delta^{1 \times 2}(\lambda \cdot B^\perp)_{B \mapsto B^* = \lambda}$$

$$\mathcal{A}_3^{(1)} = \frac{\delta^{1 \times 4}(\bar{\lambda}^\perp \cdot \bar{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2}(\lambda \cdot \bar{\lambda}) \equiv \int \frac{d^1 \times 3 W}{\text{vol}(GL_1)} \frac{\delta^{1 \times 4}(W \cdot \bar{\eta})}{\langle 1 \rangle \langle 2 \rangle \langle 3 \rangle} \delta^{1 \times 2}(W \cdot \bar{\lambda}) \delta^{2 \times 2}(\lambda \cdot W^\perp)_{W \mapsto W^*}$$
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Grassmannian Representations of Three-Point Amplitudes

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\[
1 \quad \Leftrightarrow \quad B = \left( \begin{array}{ccc}
    b_1 & b_2 & b_3 \\
    b_1' & b_2' & b_3'
\end{array} \right)
\]

\[
1 \quad \Leftrightarrow \quad W = \left( \begin{array}{c}
    w_1 \\
    w_1' \\
    w_3'
\end{array} \right)
\]

\[
\mathcal{A}_3^{(2)} = \frac{\delta^{2\times4}(\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2\times2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^2 B}{\text{vol}(GL_2)} \frac{\delta^{2\times4}(B \cdot \tilde{\eta})}{(12)(23)(31)} \delta^{2\times2}(B \cdot \tilde{\lambda}) \delta^{1\times2}(\lambda \cdot B^\perp)
\]

\[
\mathcal{A}_3^{(1)} = \frac{\delta^{1\times4}(\tilde{\lambda} \cdot \tilde{\eta})}{[12][23][31]} \delta^{2\times2}(\lambda \cdot \tilde{\lambda}) \equiv \int \frac{d^1 W}{\text{vol}(GL_1)} \frac{\delta^{1\times4}(W \cdot \tilde{\eta})}{(1)(2)(3)} \delta^{1\times2}(W \cdot \tilde{\lambda}) \delta^{2\times2}(\lambda \cdot W^\perp)
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Constructing the Correspondence: Amalgamations & Bridges
Constructing the Correspondence: Amalgamations & Bridges

**Direct/Outer Products**

\[(f_1, f_2) \mapsto f_1 \times f_2\]
\[(C_1, C_2) \mapsto C_1 \oplus C_2 \subset G(k_1 + k_2, n_1 + n_2)\]
\[(\Omega_1, \Omega_2) \mapsto \Omega_1 \wedge \Omega_2 \quad (d_1, d_2) \mapsto d_1 + d_2\]
Constructing the Correspondence: Amalgamations & Bridges

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\((\Omega_1, \Omega_2) \mapsto \Omega_1 \wedge \Omega_2 \quad (d_1, d_2) \mapsto d_1 + d_2\)

\[
\begin{pmatrix}
1 & 2 & I \\
1 & w_2 & w_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & b_4^1 \\
0 & 1 & b_4^2
\end{pmatrix}
\]
The Amalgamation of On-Shell Diagrams
Building-Up the Grassmannian Correspondence: On-Shell Varieties
The Classification of On-Shell (Cluster) Varieties

Grassmannian Representations of On-Shell Functions
Iterative Construction of Grassmannian ‘On-Shell’ Varieties
Characteristics of Grassmannian Representations

Constructing the Correspondence: Amalgamations & Bridges

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\[(\Omega_1, \Omega_2) \mapsto \Omega_1 \wedge \Omega_2 \quad (d_1, d_2) \mapsto d_1 + d_2\]

\[
\begin{pmatrix}
1 & 2 & \mathbb{1} \\
1 & w_2 & w_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & b_4^1 \\
0 & 1 & b_4^2
\end{pmatrix}
\]
Constructing the Correspondence: Amalgamations & Bridges

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\[
\begin{align*}
(f_1, f_2) & \mapsto f_1 \times f_2 \\
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Constructing the Correspondence: Amalgamations & Bridges

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(\Omega_1, \Omega_2) & \mapsto \Omega_1 \wedge \Omega_2 \quad (d_1, d_2) \mapsto d_1 + d_2
\end{align*}
\]

\[
\begin{pmatrix}
I \\
1 & 2 & 1 \\
1 & w_2 & w_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
I' \\
3 & 4 \\
1 & 0 & b_4^1 \\
0 & 1 & b_4^2
\end{pmatrix}
\]
Constructing the Correspondence: Amalgamations & Bridges

Direct/Outer Products

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\[
\begin{pmatrix}
1 & 2 & 1 \\
1 & w_2 & w_1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & b_4^1 \\
0 & 1 & b_4^2
\end{pmatrix}
\]
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Part III: Stratifying On-Shell Cluster Varieties
Constructing the Correspondence: Amalgamations & Bridges

Direct/Outer Products

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Constructing the Correspondence: Amalgamations & Bridges

Direct/Outer Products

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- \((\Omega_1, \Omega_2) \mapsto \Omega_1 \wedge \Omega_2 \ (d_1, d_2) \mapsto d_1 + d_2\)

\[
\begin{pmatrix}
1 & 2 & I & 1' \\
1 & w_2 & w_I & 3 \\
1 & w_I & w_2 & 4 \\
\end{pmatrix}
\]

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Part III: Stratifying On-Shell Cluster Varieties
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\[(\Omega_1, \Omega_2) \mapsto \Omega_1 \wedge \Omega_2 \quad (d_1, d_2) \mapsto d_1 + d_2\]

\[
C \equiv \begin{pmatrix}
1 & w_2 & w_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & b^1_4 \\
0 & 0 & 0 & 0 & 1 & b^2_4
\end{pmatrix}
\]
Constructing the Correspondence: Amalgamations & Bridges

Direct/Outer Products

\[(f_1, f_2) \mapsto f_1 \times f_2\]
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\[
C \equiv \begin{pmatrix}
1 & w_2 & w_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & b_4^1 \\
0 & 0 & 0 & 0 & 1 & b_4^2 \\
\end{pmatrix}
\]
Constructing the Correspondence: Amalgamations & Bridges

Direct/Outer Products

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\[
C \equiv \begin{pmatrix}
1 & w_2 & w_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & b^1_4 \\
0 & 0 & 0 & 0 & 1 & b^2_4
\end{pmatrix}
\]
Constructing the Correspondence: Amalgamations & Bridges

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- \((\Omega_1, \Omega_2) \mapsto \Omega_1 \wedge \Omega_2\)
  \((d_1, d_2) \mapsto d_1 + d_2\)

**Amalgamation: Gluing Legs \((A, B)\)**

- \(f \mapsto f'\)
- \(c_i \mapsto c_i \cap (c_A + c_B)^{\perp}\)
- \(C \mapsto C/(c_A + c_B) \subset G(k-1, n-2)\)
- \(\Omega \mapsto \Omega/\text{vol}(GL(1))\)
- \(d \mapsto d - 1\)

\[
C \equiv \begin{pmatrix}
1 & w_2 & w_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & b^1_4 \\
0 & 0 & 0 & 0 & 1 & b^2_4 \\
\end{pmatrix}
\]
Constructing the Correspondence: Amalgamations & Bridges

Direct/Outer Products

\[(f_1, f_2) \mapsto f_1 \times f_2\]
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\[C \equiv \begin{pmatrix}
1 & w_2 & w_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & b_4^1 \\
0 & 0 & 0 & 0 & 1 & b_4^2
\end{pmatrix}\]
Constructing the Correspondence: Amalgamations & Bridges

Direct/Outer Products

\((f_1, f_2) \mapsto f_1 \times f_2\)

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\(f \mapsto f'\)

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\(C \mapsto C / (c_A + c_B) \subset G(k - 1, n - 2)\)

\(\Omega \mapsto \Omega / \text{vol}(GL(1))\)

\(d \mapsto d - 1\)
Constructing the Correspondence: Amalgamations & Bridges

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**Amalgamation: Gluing Legs \((A, B)\)**

\[f \mapsto f' \quad c_i \mapsto c_i \cap (c_A + c_B) \perp\]

\[C \mapsto C / (c_A + c_B) \subset G(k-1, n-2)\]

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\[C \equiv \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & w_2 & 0 & b_4^1 \\ 0 & 0 & 1 & b_4^2 \end{pmatrix}\]

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Constructing the Correspondence: Amalgamations & Bridges

Direct/Outer Products

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Constructing the Correspondence: Amalgamations & Bridges

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Constructing the Correspondence: Amalgamations & Bridges

### Direct/Outer Products

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Constructing the Correspondence: Amalgamations & Bridges

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Constructing the Correspondence: Amalgamations & Bridges

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The Amalgamation of On-Shell Diagrams
Building-Up the Grassmannian Correspondence: On-Shell Varieties
The Classification of On-Shell (Cluster) Varieties
Grassmannian Representations of On-Shell Functions
Iterative Construction of Grassmannian ‘On-Shell’ Varieties
Characteristics of Grassmannian Representations

Constructing the Correspondence: Amalgamations & Bridges

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Constructing the Correspondence: Amalgamations & Bridges

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Constructing the Correspondence: Amalgamations & Bridges

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Constructing the Correspondence: Amalgamations & Bridges

Direct/Outer Products

(f₁, f₂) ↦→ f₁ × f₂
(C₁, C₂) ↦→ C₁ ⊕ C₂ ⊂ G(k₁ + k₂, n₁ + n₂)
(Ω₁, Ω₂) ↦→ Ω₁ ∧ Ω₂ (d₁, d₂) ↦→ d₁ + d₂

Amalgamation: Gluing Legs (A, B)

f ↦→ f'
cᵢ ↦→ cᵢ ∩ (c_A + c_B)⊥
C ↦→ C/(c_A + c_B) ⊂ G(k−1, n−2)
Ω ↦→ Ω/ vol(GL(1)) d ↦→ d−1

C ≡ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \hline 1 & w₂ & 0 & b₄² \\ 0 & 0 & 1 & b₄² \end{pmatrix}

f_Γ ≡ ∫ Ω_C δ^{k×2} (C·\tilde{\lambda}) δ^{2×(n−k)} (\lambda·C⊥)
Constructing the Correspondence: Amalgamations & Bridges

Direct/Outer Products

\[(f_1, f_2) \mapsto f_1 \times f_2 \]
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Constructing the Correspondence: Amalgamations & Bridges

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Constructing the Correspondence: Amalgamations & Bridges

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**Part III: Stratifying On-Shell Cluster Varieties**
Constructing the Correspondence: Amalgamations & Bridges

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Constructing the Correspondence: Amalgamations & Bridges

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## Constructing the Correspondence: Amalgamations & Bridges

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### Adding a ‘Bridge’ to Legs \((A, B)\)

\[f \mapsto f' \quad c_B \mapsto c_B + \alpha c_A\]
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Constructing the Correspondence: Amalgamations & Bridges

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Constructing the Correspondence: Amalgamations & Bridges

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Constructing the Correspondence: Amalgamations & Bridges

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Constructing the Correspondence: Amalgamations & Bridges

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Constructing the Correspondence: Amalgamations & Bridges

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**Part III: Stratifying On-Shell Cluster Varieties**
Constructing the Correspondence: Amalgamations & Bridges

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Constructing the Correspondence: Amalgamations & Bridges

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Construction via ‘Boundary Measurements’

A more direct way to construct $C(\alpha)$ is via boundary measurements:
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A more direct way to construct $C(\alpha)$ is via boundary measurements:

$$\Omega \equiv d\alpha_1 \wedge \cdots \wedge d\alpha_9 \times \det(\text{Adj}_{N-4}) C(\alpha) \equiv \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & \alpha_5 (1+\alpha_8) \alpha_2 \alpha_6 \alpha_7 \alpha_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & \alpha_5 & \alpha_1 & \alpha_2 & \alpha_4 & \alpha_4 & \alpha_7 & 0 & 1 & 0 & 0 & 0 \\ \alpha_5 & \alpha_9 & \alpha_3 & \alpha_4 & \alpha_6 & \alpha_9 & (\alpha_3 \alpha_4 + \alpha_6 \alpha_9) & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$
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$$
\Omega \equiv \prod_{i=1}^{9} \alpha_i \wedge \cdots \wedge \prod_{i=4}^{9} \det(\text{Adj} N_{\alpha_i}) 
$$

$$
C(\alpha) \equiv \begin{pmatrix}
    c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\
    \alpha_5 & \alpha_2 & \alpha_6 & \alpha_1 & \alpha_7 & \alpha_8 \\
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  \alpha_1 & \alpha_5 & \alpha_1 & \alpha_2 + \alpha_4 & \alpha_4 & \alpha_7 & 0 \\
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\alpha_1 & \alpha_5 & \alpha_1 & \alpha_2 + \alpha_4 & \alpha_4 & \alpha_7 & 0 & 1 & 0 \\
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A more direct way to construct $C(\alpha)$ is via **boundary measurements**:

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$$C(\alpha) \equiv \begin{pmatrix}
    c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\
    \alpha_5(1+\alpha_8) & \alpha_2 & \alpha_6 & \alpha_7 & \alpha_8 & 1 & 0 & 0 \\
    \alpha_1 & \alpha_5 & \alpha_1 & \alpha_2+\alpha_4 & \alpha_4 & \alpha_7 & 0 & 1 & 0 \\
    \alpha_5 & \alpha_9 & \alpha_3 & \alpha_4 & \alpha_7(\alpha_3\alpha_4+\alpha_6\alpha_9) & 0 & 0 & 1
\end{pmatrix}$$
Construction via ‘Boundary Measurements’

A more direct way to construct \( C(\alpha) \) is via boundary measurements:

\[
\Omega \equiv \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_9}{\alpha_9} \times \det(1-\text{Adj})_{N-4}
\]

\[
C(\alpha) \equiv \begin{pmatrix}
\alpha_5(1+\alpha_8) & \alpha_2 & \alpha_6 & \alpha_7 & \alpha_8 & 1 & 0 & 0 \\
\alpha_1 & \alpha_5 & \alpha_1 & \alpha_2 + \alpha_4 & \alpha_4 & \alpha_7 & 0 & 1 & 0 \\
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\]
General Characteristics of the Correspondence

\[ f_\Gamma \equiv \int \Omega_C \delta^{k \times 2} (C \cdot \tilde{\lambda}) \delta^{2 \times (n-k)} (\lambda \cdot C^\perp) \]
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General Characteristics

\[ n: \text{the number of external legs} \]

\[ k: \text{the number of 'sources': } 2 \]

\[ n_B + n_W - n_I \text{ (trivalent)} \]

\[ d: \text{the number of coordinates} \]

\[ C(\vec{\alpha}): 2n_V - n_I \text{ (trivalent); } n + n_I - n_V \text{ (general)} \]

\[ \text{number of } \delta\text{-functions (beyond momentum conservation) is always: } 2n - 4 \]

(notice that when \( k = 2 \) (MHV), the constraints always require that \( C \rightarrow C^* = \lambda \))

\[ \text{recall that } \dim(G(k, n)) = k(n - \frac{k(n-k)}{2}); \]

and so if \( d > k(n - \frac{k(n-k)}{2}) \), some of the coordinates must be degenerate

Definition: a diagram is called reduced if \( d(\Gamma) = \dim(C) \)

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For $k = 2$ and $\hat{n}_\delta = 0$, reduced diagrams correspond to \emph{top-dimensional} varieties.
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$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 6 \\ 4 & 5 & 1 \end{pmatrix}$$
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$$
\begin{align*}
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\[
\begin{cases}
(1 2 3) \\
(2 5 6) \\
(3 4 6) \\
(4 5 1)
\end{cases}
\rightarrow
\]
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\[ \implies C^\perp (\vec{\alpha}^*) \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \]
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1 & 2 & 3 & 4 & 5 & 6 \\
\langle 23 \rangle & \langle 31 \rangle & \langle 12 \rangle & 0 & 0 & 0
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$$
\begin{aligned}
(1 & 2 & 3) \\
(2 & 5 & 6) \\
(3 & 4 & 6) \\
(4 & 5 & 1)
\end{aligned}
\Rightarrow
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
\langle 23 \rangle \langle 31 \rangle \langle 12 \rangle & 0 & 0 & 0 \\
0 & \langle 56 \rangle & 0 & 0 & \langle 62 \rangle \langle 25 \rangle
\end{pmatrix}
\equiv
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0 & 0 & \langle 46 \rangle & \langle 63 \rangle & 0 & \langle 34 \rangle \\
\langle 45 \rangle & 0 & 0 & \langle 51 \rangle & \langle 14 \rangle & 0
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\end{pmatrix} \\
\end{align*}
\]

\[
f_\Gamma \equiv \frac{1}{\langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \langle 56 \rangle \langle 62 \rangle \langle 25 \rangle \langle 46 \rangle \langle 63 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle \langle 14 \rangle} \delta^2 \times 4 (C^* \vec{\eta}) \delta^2 \times 2 (C^* \vec{\lambda})
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$$

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Application: Classifying On-Shell Functions for $k = 2$ (MHV)

For $k = 2$ and $\hat{n}_\delta = 0$, reduced diagrams correspond to *top-dimensional* varieties.

A simple exercise shows that for any such reduced diagram:
- $n_B = (n-2)$
- and each blue vertex must connect to exactly *three* external legs
  —through (arbitrary-length) chains of white vertices

We may label such diagrams by the triples, $\tau$, of legs attached to blue vertices:

\[
\begin{align*}
\tau & = \{ (1\ 2\ 3), (2\ 5\ 6), (3\ 4\ 6), (4\ 5\ 1) \} \\
\Rightarrow & \quad C^\perp(\vec{\alpha}^*) \equiv \\
& \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
\langle 23 \rangle & \langle 31 \rangle & \langle 12 \rangle & 0 & 0 & 0 \\
0 & \langle 56 \rangle & 0 & 0 & \langle 62 \rangle & \langle 25 \rangle \\
0 & 0 & \langle 46 \rangle & \langle 63 \rangle & 0 & \langle 34 \rangle \\
\langle 45 \rangle & 0 & 0 & \langle 51 \rangle & \langle 14 \rangle & 0
\end{pmatrix}
\end{align*}
\]

\[
f_\Gamma \equiv \frac{1}{\langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \langle 56 \rangle \langle 62 \rangle \langle 25 \rangle \langle 46 \rangle \langle 63 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle \langle 14 \rangle} \delta^{2 \times 4}(C^*\vec{\eta}) \delta^{2 \times 2}(C^*\vec{\lambda})
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\[
\begin{aligned}
C^{-1}(\vec{\alpha}^*) &\equiv \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
\langle 23 \rangle & \langle 31 \rangle & \langle 12 \rangle & 0 & 0 & 0 \\
0 & \langle 56 \rangle & 0 & 0 & \langle 62 \rangle & \langle 25 \rangle \\
0 & 0 & \langle 46 \rangle & \langle 63 \rangle & 0 & \langle 34 \rangle \\
\langle 45 \rangle & 0 & 0 & \langle 51 \rangle & \langle 14 \rangle & 0
\end{pmatrix} \\
\end{aligned}
\]

\[
f_\Gamma \equiv \frac{1}{\langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \langle 56 \rangle \langle 62 \rangle \langle 25 \rangle \langle 46 \rangle \langle 63 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle \langle 14 \rangle} \delta^{2 \times 4}(C^*:\widetilde{\eta}) \delta^{2 \times 2}(C^*:\widetilde{\lambda})
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We may label such diagrams by the triples, $\tau$, of legs attached to blue vertices:

$$\{ (1\, 2\, 3), (2\, 5\, 6), (3\, 4\, 6), (4\, 5\, 1) \} \implies C_{\perp}(\vec{\alpha}^{*}) \equiv \left( \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\langle 23 \rangle \langle 31 \rangle \langle 12 \rangle & 0 & 0 & 0 & 0 & 0 \\
0 & \langle 56 \rangle & 0 & 0 & \langle 62 \rangle & \langle 25 \rangle \\
0 & 0 & \langle 46 \rangle & \langle 63 \rangle & 0 & \langle 34 \rangle \\
\langle 45 \rangle & 0 & 0 & \langle 51 \rangle & \langle 14 \rangle & 0
\end{array} \right)$$

$$f_\Gamma \equiv \frac{\left( \langle 34 \rangle \langle 51 \rangle \langle 62 \rangle + \langle 14 \rangle \langle 25 \rangle \langle 63 \rangle \right)^2}{\langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \langle 56 \rangle \langle 62 \rangle \langle 25 \rangle \langle 46 \rangle \langle 63 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle \langle 14 \rangle} \delta^{2 \times 4} (\lambda \cdot \vec{\eta}) \delta^{2 \times 2} (\lambda \cdot \vec{\lambda})$$
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$$\begin{align*}
(1 & 2 3) \\
(2 & 5 6) \\
(3 & 4 6) \\
(4 & 5 1)
\end{align*}$$

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\end{pmatrix}
\end{align*}$$

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The Parke-Taylor formula for MHV amplitudes can be interpreted geometrically as imposing a certain kind of ‘positivity’ among the \( \lambda_a \) variables:

\[
\text{PT}(1,2,3,4,5,6) \equiv \delta_{24} \times 4 \left( \lambda \cdot \tilde{\eta} \right) \\
\delta_{12} \times 2 \left( \lambda \cdot \tilde{\lambda} \right) \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle \\
\equiv \{ (1,3,4) (3,2,4) \} \equiv \begin{cases} \\
\{ \end{cases}
\]

\[
\tilde{f}_\Gamma = \sum \{ \sigma \in (S_n/\mathbb{Z}_n) | \forall \tau \in T : \sigma \tau_1 < \sigma \tau_2 < \sigma \tau_3 \} \left( \text{PT}(\sigma_1,\ldots,\sigma_n), \ldots \right).
\]
Extended ‘Positivity’ and Parke-Taylor Completeness

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$$

\[\iff\]

Diagram: Points 1 through 6 connected in a cycle.
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1 \\
2 \\
3 \\
4 \\
5 \\
6
\end{array} $$

$$ \Gamma = \sum \left\{ \sigma \in \left( S_n / \mathbb{Z}_n \right) \mid \forall \tau \in T: \sigma \tau \leq \sigma \right\} $$

$$ \text{PT}(\sigma_1, \ldots, \sigma_n), $$
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1 \\
2 \\
3 \\
4 \\
5 \\
6
\end{array}$$

$$\text{PT}(1, 3, 2, 4) \iff \begin{Bmatrix}
(1 \ 3 \ 4) \\
(3 \ 2 \ 4)
\end{Bmatrix} \iff \begin{Bmatrix}
1 \\
2 \\
3 \\
4
\end{Bmatrix}$$
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$$\left\{ (1 \ 3 \ 4) \right\} \iff \left\{ \begin{array}{c} 1 \\ 3 \\ 2 \\ 4 \\ \end{array} \right\}$$

$$PT(1, 3, 2, 4)$$
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Extended ‘Positivity’ and Parke-Taylor Completeness

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$\left\{ (1 \ 4 \ 3) \right\} \iff \left\{ \begin{array}{c} 1 \\ 4 \\ 3 \\ 2 \\ 1 \\ 2 \\ 3 \end{array} \right\}$

$\zeta f_\Gamma = \sum \left\{ \sigma \in \left( S_n / \mathbb{Z}_n \right) \right\}$

$$\text{PT}(\sigma_1, ..., \sigma_n)$$

or

$$\text{PT}(1, 4, 3, 2), \text{ PT}(1, 2, 4, 3)$$
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$$\left\{ \begin{array}{c}
(1) \ 4 \ 3 \\
(3) \ 2 \ 4
\end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c}
1 \quad 4 \quad 3 \\
2 \quad 3
\end{array} \right\} \text{ or } \left\{ \begin{array}{c}
1 \quad 4 \\
2 \quad 3
\end{array} \right\} \text{ or } \left\{ \begin{array}{c}
1 \quad 2 \quad 4 \quad 3
\end{array} \right\}$$
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1 \\
2 \\
3 \\
5 \\
4 \\
6
\end{array}$$

$$\tilde{f}_\Gamma = \sum_{\{\sigma \in (S_n/Z_n)\mid \forall \tau \in T: \sigma_{\tau_1} < \sigma_{\tau_2} < \sigma_{\tau_3}\}} PT(\sigma_1, \ldots, \sigma_n)$$
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1 \\
2 \\
3 \\
4 \\
5 \\
6
\end{array}
\]

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$$\sim \begin{cases} \langle 1 4 3 \rangle \langle 2 3 4 \rangle \langle 3 2 4 \rangle \langle 1 4 3 \rangle \langle 2 3 4 \rangle & \text{or} \\ \frac{1}{2} \left( \langle 1 4 3 \rangle \langle 2 3 4 \rangle + \langle 2 3 4 \rangle \langle 3 2 4 \rangle \right) & \end{cases}$$

$$\tilde{f}_\Gamma = \sum_{\{\sigma \in (S_n/\mathbb{Z}_n) | \forall \tau \in T: \sigma_{\tau_1} < \sigma_{\tau_2} < \sigma_{\tau_3} \}} PT(\sigma_1, \ldots, \sigma_n)$$
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$$

$$
\tilde{f}_\Gamma = \sum \text{PT}(\sigma_1, \ldots, \sigma_n),
$$

$$
\{ \sigma \in (S_n/Z_n) | \forall \tau \in T: \sigma_{\tau_1} < \sigma_{\tau_2} < \sigma_{\tau_3} \}$$
Geometry of Kleiss-Kuijf Relations and $U(1)$-Decoupling

This gives a geometric interpretation of the $U(1)$-decoupling and KK-relations:
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This gives a geometric interpretation of the $U(1)$-decoupling and KK-relations:

\[
PT(1, 2, \ldots, n) = \sum_{\sigma} PT(1, \sigma_1, \ldots, \sigma_{n-2}, n).
\]

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Geometry of Kleiss-Kuijf Relations and $U(1)$-Decoupling

This gives a geometric interpretation of the $U(1)$-decoupling and KK-relations:

$$\text{PT}(1, 2, \ldots, n)$$

$$(2\ n\ 1)$$
$$(2\ 3\ 4)$$
$$(2\ 4\ 5)$$
$$\vdots$$
$$(2\ n-1\ n)$$

$$(2\ 1\ n)$$
$$(2\ 3\ 4)$$
$$(2\ 4\ 5)$$
$$\vdots$$
$$(2\ n-1\ n)$$

$$\text{PT}(1, n, 2, \ldots, n-1) + \ldots + \text{PT}(1, 3, \ldots, n, 2)$$
Geometry of Kleiss-Kuijf Relations and $U(1)$-Decoupling

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\[
\begin{align*}
\begin{pmatrix}
(2 & n & 1) \\
(2 & 3 & 4) \\
(2 & 4 & 5) \\
\vdots \\
(2n-1 & n)
\end{pmatrix} & = \\
\text{PT}(1, 2, \ldots, n) & \text{PT}(1, n, 2, \ldots, n-1) + \ldots + \text{PT}(1, 3, \ldots, n, 2)
\end{align*}
\]
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Part III: Stratifying On-Shell Cluster Varieties
Geometry of Kleiss-Kuijf Relations and $U(1)$-Decoupling

This gives a geometric interpretation of the $U(1)$-decoupling and KK-relations:

\[
\begin{align*}
\frac{1}{n} \left\{ \begin{array}{c}
(2 & n 1) \\
(2 & 3 4) \\
\vdots \\
(2 & n-1 n)
\end{array} \right\} &= \left\{ \begin{array}{c}
(2 & 1 n) \\
(2 & 3 4) \\
\vdots \\
(2 & n-1 n)
\end{array} \right\} \\
- \text{PT}(1, 2, \ldots, n) &= \text{PT}(1, n, 2, \ldots, n-1) + \ldots + \text{PT}(1, 3, \ldots, n, 2)
\end{align*}
\]
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This gives a geometric interpretation of the $U(1)$-decoupling and KK-relations:
Geometry of Kleiss-Kuijf Relations and $U(1)$-Decoupling

This gives a geometric interpretation of the $U(1)$-decoupling and KK-relations:

\[
(\alpha_1 \cdots \alpha_{-2} \alpha_{-1} n) (\beta_1 \cdots \beta_{-2} \beta_{-1}) = \sum_{\sigma} \text{PT}(1, \sigma_1, \ldots, \sigma_n - 2, n).
\]
Geometry of Kleiss-Kuijf Relations and $U(1)$-Decoupling

This gives a geometric interpretation of the $U(1)$-decoupling and KK-relations:

\[
\begin{pmatrix}
(1 \alpha_1 n) \\
(\alpha_1 \alpha_2 n) \\
\vdots \\
(\alpha_{-2} \alpha_{-1} n) \\
(n \beta_1 \beta_2) \\
\vdots \\
(n \beta_{-2} \beta_{-1}) \\
(n \beta_{-1} 1)
\end{pmatrix}
\]

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Part III: Stratifying On-Shell Cluster Varieties
Geometry of Kleiss-Kuijf Relations and $U(1)$-Decoupling

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\[
\begin{align*}
\left\{ \begin{array}{c}
(1 \alpha_1 \ n) \\
(\alpha_1 \alpha_2 \ n) \\
\vdots \\
(\alpha_{-2} \alpha_{-1} \ n) \\
(n \beta_1 \beta_2) \\
\vdots \\
(n \beta_{-2} \beta_{-1}) \\
(n \beta_{-1} \ 1) \\
\end{array} \right\} 
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{c}
(1 \alpha_1 \ n) \\
(\alpha_1 \alpha_2 \ n) \\
\vdots \\
(\alpha_{-2} \alpha_{-1} \ n) \\
(n \beta_1 \beta_2) \\
\vdots \\
(n \beta_{-2} \beta_{-1}) \\
(n \beta_{-1} \ 1) \\
\end{array} \right\} 
\end{align*}
\]

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Part III: Stratifying On-Shell Cluster Varieties
Geometry of Kleiss-Kuijf Relations and $U(1)$-Decoupling

This gives a geometric interpretation of the $U(1)$-decoupling and KK-relations:

$$(-1)^{n_{\beta}} \left\{ \begin{array}{c}
(1 \alpha \_1 n) \\
(\alpha \_1 \alpha \_2 n) \\
\vdots \\
(\alpha \_2 \alpha \_1 n) \\
(n \beta \_1 \beta \_2) \\
\vdots \\
(n \beta \_2 \beta \_1) \\
(n \beta \_1 1) \\
(n 1 \beta \_1) \\
\end{array} \right\} = \left\{ \begin{array}{c}
(1 \alpha \_1 n) \\
(\alpha \_1 \alpha \_2 n) \\
\vdots \\
(\alpha \_2 \alpha \_1 n) \\
(n \beta \_1 \beta \_2) \\
\vdots \\
(n \beta \_2 \beta \_1) \\
(n 1 \beta \_1) \\
\end{array} \right\}$$
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\[
(-1)^{n_\beta} \times \text{PT}(1, \alpha_1, \ldots, \alpha_{-1}, n, \beta_1, \ldots, \beta_{-1}) = \sum_{\sigma} \text{PT}(1, \sigma_1, \ldots, \sigma_{n-2}, n),
\]

where $\sigma \in (\{\alpha_1, \ldots, \alpha_{-1}\} \uplus \{\beta_1, \ldots, \beta_{-1}\})$. 

\[
\begin{align*}
(-1)^{n_\beta} \times & \quad \begin{pmatrix}
(1 & \alpha_1 & n) \\
(\alpha_1 & \alpha_2 & n) \\
& \vdots \\
(\alpha_{-2} & \alpha_{-1} & n) \\
(n & \beta_1 & \beta_2) \\
& \vdots \\
(n & \beta_{-2} & \beta_{-1}) \\
(n & \beta_{-1} & 1)
\end{pmatrix} = \\
& \quad \begin{pmatrix}
(1 & \alpha_1 & n) \\
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& \vdots \\
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\[
(-1)^{n_\beta} = \sum \begin{pmatrix} (1 \alpha_1 n) \\ (\alpha_1 \alpha_2 n) \\ \vdots \\ (\alpha_{-2} \alpha_{-1} n) \\ (n \beta_1 \beta_2) \\ \vdots \\ (n \beta_{-2} \beta_{-1}) \\ (n \beta_{-1} 1) \end{pmatrix} \quad = \quad \begin{pmatrix} (1 \alpha_1 n) \\ (\alpha_1 \alpha_2 n) \\ \vdots \\ (\alpha_{-2} \alpha_{-1} n) \\ (n \beta_2 \beta_1) \\ \vdots \\ (n \beta_{-1} \beta_{-2}) \\ (n 1 \beta_{-1}) \end{pmatrix}
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Definitions:
A diagram is reduced if \( \dim(C) = d(\Gamma) \).
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Conjectures:
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Part III: Stratifying On-Shell Cluster Varieties
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\partial \begin{pmatrix}
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\[ \partial \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \Rightarrow \begin{cases} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \end{cases} \]
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(1)(2)(3)(4)
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Amplitudes 2018 Summer School  QMAP, University of California, Davis  **Part III: Stratifying On-Shell Cluster Varieties**
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- 24 (equivalence classes of) top-dimensional cells

<table>
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<th>Equivalence Classes</th>
<th>Planar</th>
<th>Prime</th>
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<td>3</td>
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<td>7</td>
<td>6</td>
<td>3</td>
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  - 1 is planar

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Classification of On-Shell Varieties for 6-Point NMHV ($k = 3$)

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### Summary of the Classification of On-Shell Varieties of $G(3,6)$

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The Classification of Top-Dim On-Shell Varieties of $G(3,6)$
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1 2 3 4 5 6
1 2 3 4 5 6
1 2 3 4 5 6
1 2 3 4 5 6
1 2 3 4 5 6
1 2 3 4 5 6
1 2 3 4 5 6
1 2 3 4 5 6

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[Diagram of topologically distinct varieties]
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**Part III:** Stratifying On-Shell Cluster Varieties
Enumeration of All (ten) ‘Leading Singularities’ of $G(3,6)$
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**Enumeration of All (ten) ‘Leading Singularities’ of $G(3,6)$**

\[
\begin{aligned}
f_1 &\equiv \oint \Omega_1 = \frac{\delta^3 \times 4 (C^* \cdot \tilde{\eta}) \delta^2 \times 2 (\lambda \cdot \tilde{\lambda})}{(234)(345)(456)(561)(612)}_{C^*} \\
&= \frac{\delta^3 \times 4 (C^* \cdot \tilde{\eta}) \delta^2 \times 2 (\lambda \cdot \tilde{\lambda})}{\langle 23 \rangle [56] \langle 3|4+5|6]s_{456} \langle 1|5+6|4\rangle [12] [45]}
\end{aligned}
\]

\[
C^* \equiv \begin{pmatrix}
\lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\
\lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
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**Enumeration of All (ten) ‘Leading Singularities’ of $G(3,6)$**

\[
f_1 \equiv \oint \Omega_1 = \left. \frac{\delta^{3\times4}(C^* \cdot \tilde{\eta}) \delta^{2\times2}(\lambda \cdot \tilde{\lambda})}{(234)(345)(456)(561)(612)} \right|_{C^*} \]
\[
= \langle 23 \rangle [56] \langle 3|4+5|6 \rangle s_{456} \langle 1|5+6|4 \rangle [12] [45]
\]

\[
f_2 \equiv \oint \Omega_2 = \left. \frac{(235) \delta^{3\times4}(C^* \cdot \tilde{\eta}) \delta^{2\times2}(\lambda \cdot \tilde{\lambda})}{(136)(156)(234)(245)(256)(345)} \right|_{C^*} \]
\[
= \langle 23 \rangle [64] \delta^{3\times4}(C^* \cdot \tilde{\eta}) \delta^{2\times2}(\lambda \cdot \tilde{\lambda})
\]
\[
= \langle 13 \rangle [45] \langle 1|5+6|4 \rangle \langle 23 \rangle [56] \langle 2|4+5|6 \rangle \langle 2|5+6|4 \rangle \langle 3|4+5|6 \rangle
\]

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**Part III: Stratifying On-Shell Cluster Varieties**

**Enumeration of All (ten) ‘Leading Singularities’ of \( G(3,6) \)**

\[
f_1 \equiv \oint \Omega_1 = \frac{\delta^3 \times 4(C^* \cdot \tilde{\eta}) \delta^2 \times 2(\lambda \cdot \tilde{\lambda})}{(234)(345)(456)(561)(612)} \bigg|_{C^*} \\
= \frac{\delta^3 \times 4(C^* \cdot \tilde{\eta}) \delta^2 \times 2(\lambda \cdot \tilde{\lambda})}{\langle 23 \rangle [56] \langle 3|4+5|6 \rangle s_{456} \langle 1|5+6|4 \rangle \langle 12 \rangle [45]}
\]

\[
f_2 \equiv \oint \Omega_2 = \frac{(235) \delta^3 \times 4(C^* \cdot \tilde{\eta}) \delta^2 \times 2(\lambda \cdot \tilde{\lambda})}{(136)(156)(234)(245)(256)(345)} \bigg|_{C^*} \\
= \frac{\delta^3 \times 4(C^* \cdot \tilde{\eta}) \delta^2 \times 2(\lambda \cdot \tilde{\lambda})}{\langle 13 \rangle [45] \langle 1|5+6|4 \rangle \langle 23 \rangle [56] \langle 2|4+5|6 \rangle \langle 2|5+6|4 \rangle \langle 3|4+5|6 \rangle}
\]

\[
f_3 \equiv \oint \Omega_4 = \frac{(145) \delta^3 \times 4(C^* \cdot \tilde{\eta}) \delta^2 \times 2(\lambda \cdot \tilde{\lambda})}{(124)(136)(156)(245)(345)(456)} \bigg|_{C^*} \\
= \frac{\delta^3 \times 4(C^* \cdot \tilde{\eta}) \delta^2 \times 2(\lambda \cdot \tilde{\lambda})}{\langle 1|4+5|6 \rangle s_{456}}
\]
Enumeration of All (ten) ‘Leading Singularities’ of $G(3,6)$

$$f_4 \equiv \oint_{(123)=0} \Omega_5 = \left. \frac{(135) \delta^{3 \times 4}(C^* \cdot \tilde{\eta}) \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda})}{(124)(145)(156)(236)(345)(356)} \right|_{C^*} \langle 13 \rangle [64] \delta^{3 \times 4}(C^* \cdot \tilde{\eta}) \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda})$$

$$= \langle 12 \rangle [56] \langle 1|4+5|6 \rangle \langle 1|5+6|4 \rangle [23] [45] \langle 3|4+5|6 \rangle \langle 3|5+6|4 \rangle$$

$$C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$
**Enumeration of All (ten) ‘Leading Singularities’ of $G(3,6)$**

\[
f_4 \equiv \oint \Omega_5 = \frac{(135) \delta^3 \times 4 (C^* \cdot \tilde{\eta}) \delta^2 \times 2 (\lambda \cdot \tilde{\lambda})}{(124)(145)(156)(236)(345)(356)} |_{C^*}
\]

\[
= \langle 13 \rangle [64] \delta^3 \times 4 (C^* \cdot \tilde{\eta}) \delta^2 \times 2 (\lambda \cdot \tilde{\lambda})
\]

\[
= \frac{\langle 12 \rangle [56] \langle 1|4+5|6 \rangle \langle 1|5+6|4 \rangle \langle 23 \rangle [45] \langle 3|4+5|6 \rangle \langle 3|5+6|4 \rangle}{\langle 13 \rangle [64] \delta^3 \times 4 (C^* \cdot \tilde{\eta}) \delta^2 \times 2 (\lambda \cdot \tilde{\lambda})}
\]

\[
f_5 \equiv \oint \Omega_9 = \frac{(125) \delta^3 \times 4 (C^* \cdot \tilde{\eta}) \delta^2 \times 2 (\lambda \cdot \tilde{\lambda})}{(134)(156)(245)(256)(16)(25) \cap (34)} |_{C^*}
\]

\[
= \langle 12 \rangle [64] \delta^3 \times 4 (C^* \cdot \tilde{\eta}) \delta^2 \times 2 (\lambda \cdot \tilde{\lambda})
\]

\[
= \frac{\langle 13 \rangle [56] \langle 1|5+6|4 \rangle \langle 2|4+5|6 \rangle \langle 2|5+6|4 \rangle \langle 23 \rangle [56] \langle 1|5+6|4 \rangle - \langle 12 \rangle [45] \langle 3|4+5|6 \rangle}{\langle 13 \rangle [56] \langle 1|5+6|4 \rangle \langle 2|4+5|6 \rangle \langle 2|5+6|4 \rangle \langle 23 \rangle [56] \langle 1|5+6|4 \rangle - \langle 12 \rangle [45] \langle 3|4+5|6 \rangle}
\]
**Enumeration of All (ten) ‘Leading Singularities’ of $G(3,6)$**

| $f_4 \equiv \oint \Omega_5 = \frac{(135) \delta^3 \times 4(C^* \cdot \tilde{\eta}) \delta^2 \times 2(\lambda \cdot \tilde{\lambda})}{(124)(145)(156)(236)(345)(356)} | C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 & 0 \\ \lambda_3^3 & \lambda_4^3 & \lambda_5^3 & \lambda_6^3 & 0 & 0 \\ 0 & 0 & 0 & 56 & 64 & 45 \end{pmatrix} = \langle 13 \rangle [64] \delta^3 \times 4(C^* \cdot \tilde{\eta}) \delta^2 \times 2(\lambda \cdot \tilde{\lambda}) = \langle 12 \rangle [56] \langle 14+5 | 6 \rangle \langle 15+6 | 4 \rangle \langle 23 \rangle [45] \langle 3 | 4+5 | 6 \rangle \langle 3 | 5+6 | 4 \rangle | 5+6 | 4 \rangle |

| $f_5 \equiv \oint \Omega_9 = \frac{(125) \delta^3 \times 4(C^* \cdot \tilde{\eta}) \delta^2 \times 2(\lambda \cdot \tilde{\lambda})}{(134)(156)(245)(256)(16(25) \cap (34))} | C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 & 0 \\ \lambda_3^3 & \lambda_4^3 & \lambda_5^3 & \lambda_6^3 & 0 & 0 \\ 0 & 0 & 0 & 56 & 64 & 45 \end{pmatrix} = \langle 12 \rangle [64] \delta^3 \times 4(C^* \cdot \tilde{\eta}) \delta^2 \times 2(\lambda \cdot \tilde{\lambda}) = \langle 13 \rangle [56] \langle 15+6 | 4 \rangle \langle 2 | 4+5 | 6 \rangle \langle 2 | 5+6 | 4 \rangle \langle 23 \rangle [56] \langle 1 | 5+6 | 4 \rangle \langle 1 | 5+6 | 4 \rangle - \langle 12 \rangle [45] \langle 3 | 4+5 | 6 \rangle |

| $f_6 \equiv \oint \Omega_{12} = \frac{(134)^2(456) \delta^3 \times 4(C^* \cdot \tilde{\eta}) \delta^2 \times 2(\lambda \cdot \tilde{\lambda})}{(124)(145)(146)(156)(234)(345)(346)(356)} | C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 & 0 \\ \lambda_3^3 & \lambda_4^3 & \lambda_5^3 & \lambda_6^3 & 0 & 0 \\ 0 & 0 & 0 & 56 & 64 & 45 \end{pmatrix} = \langle 13 \rangle ^2 s_{456} \delta^3 \times 4(C^* \cdot \tilde{\eta}) \delta^2 \times 2(\lambda \cdot \tilde{\lambda}) = \langle 12 \rangle \langle 1 | 4+5 | 6 \rangle \langle 1 | 4+6 | 5 \rangle \langle 1 | 5+6 | 4 \rangle \langle 23 \rangle \langle 3 | 4+5 | 6 \rangle \langle 3 | 4+6 | 5 \rangle \langle 3 | 5+6 | 4 \rangle | 5+6 | 4 \rangle | 4 \rangle |

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**Part III: Stratifying On-Shell Cluster Varieties**

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Part III: Stratifying On-Shell Cluster Varieties

**Enumeration of All (ten) ‘Leading Singularities’ of $G(3,6)$**

$$
 f_7 \equiv \oint_{(123)=0} \Omega_{13} = \left. \frac{(145)^2 \delta^3 \cdot 4 \left( C_\ast \cdot \tilde{\eta} \right) \delta^2 \cdot 2 \left( \lambda \cdot \tilde{\lambda} \right)}{(125)(134)(146)(156)(245)(345)(456)} \right|_{C^\ast} = \\
\frac{\langle 1|4+5|6 \rangle^2 \delta^3 \cdot 4 \left( C_\ast \cdot \tilde{\eta} \right) \delta^2 \cdot 2 \left( \lambda \cdot \tilde{\lambda} \right)}{\langle 12 \rangle [64] \langle 13 \rangle [56] \langle 1|4+6|5 \rangle \langle 1|5+6|4 \rangle \langle 2|4+5|6 \rangle \langle 3|4+5|6 \rangle s_{456}} \\
$$

$$
 C^\ast \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix} 
$$
Enumeration of All (ten) ‘Leading Singularities’ of $G(3,6)$

\[ f_7 \equiv \oint \Omega_{13} = \frac{(145)^2 \delta^3 \times 4 \left( C^* \cdot \tilde{\eta} \right) \delta^2 \times 2 \left( \lambda \cdot \tilde{\lambda} \right)}{(125)(134)(146)(156)(245)(345)(456)} \bigg|_{C^*} \]

\[ = \frac{\langle 1 \mid 4+5 \mid 6 \rangle^2 \delta^3 \times 4 \left( C^* \cdot \tilde{\eta} \right) \delta^2 \times 2 \left( \lambda \cdot \tilde{\lambda} \right)}{\langle 12 \rangle [64] \langle 13 \rangle [56] \langle 1 \mid 4+5 \mid 6 \rangle \langle 1 \mid 5+6 \mid 4 \rangle \langle 2 \mid 4+5 \mid 6 \rangle [3 \mid 4+5 \mid 6 \rangle} s_{456} \]

\[ f_8 \equiv \oint \Omega_{16} = \int \frac{d\alpha_1}{\alpha_1} \ldots \frac{d\alpha_8}{\alpha_8} \delta^3 \times 4 \left( C(\alpha) \cdot \tilde{\eta} \right) \delta^3 \times 2 \left( C(\alpha) \cdot \tilde{\lambda} \right) \delta^2 \times 3 \left( \lambda \cdot C^\perp(\alpha) \right) \]

\[ C(\alpha) \equiv \begin{pmatrix} 1 & \alpha_6 & \alpha_6 & \alpha_7 & 0 & 0 & \alpha_1 \\ 0 & 1 & \alpha_5 + \alpha_7 & 0 & \alpha_2 & \alpha_2 & \alpha_4 \\ \alpha_8 & 0 & 0 & 1 & \alpha_3 & \alpha_3 & \alpha_4 \end{pmatrix} \]
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### Enumeration of All (ten) ‘Leading Singularities’ of $G(3,6)$

\[
f_7 \equiv \oint \Omega_{13} = \frac{(145)^2 \delta^3 \times 4(C^* \cdot \tilde{\eta}) \delta^2 \times 2(\lambda \cdot \tilde{\lambda})}{(125)(134)(146)(156)(245)(345)(456)} \bigg|_{C^*} = \frac{\langle 1|4+5|6 \rangle^2 \delta^3 \times 4(C^* \cdot \tilde{\eta}) \delta^2 \times 2(\lambda \cdot \tilde{\lambda})}{\langle 12 \rangle \langle 64 \rangle \langle 13 \rangle \langle 56 \rangle \langle 1|4+6|5 \rangle \langle 15+6|4 \rangle \langle 24+5|6 \rangle \langle 34+5|6 \rangle s_{456}}
\]

\[
f_8 \equiv \oint \Omega_{16} = \int \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_8}{\alpha_8} \delta^3 \times 4(C(\alpha) \cdot \tilde{\eta}) \delta^3 \times 2(C(\alpha) \cdot \tilde{\lambda}) \delta^2 \times 3(\lambda \cdot C^\perp(\alpha))
\]

\[
C(\alpha) \equiv \begin{pmatrix}
1 & \alpha_6 & \alpha_7 & 0 & 0 & \alpha_1 \\
0 & 1 & \alpha_5+\alpha_7 & 0 & \alpha_2 & \alpha_2 \alpha_4 \\
\alpha_8 & 0 & 0 & 1 & \alpha_3 & \alpha_3 \alpha_4
\end{pmatrix}
\]

\[
f_9 \equiv \oint \Omega_{18} = \int \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_8}{\alpha_8} \delta^3 \times 4(C(\alpha) \cdot \tilde{\eta}) \delta^3 \times 2(C(\alpha) \cdot \tilde{\lambda}) \delta^2 \times 3(\lambda \cdot C^\perp(\alpha))
\]

\[
C(\alpha) \equiv \begin{pmatrix}
1 & \alpha_5 & \alpha_7 & 0 & 0 & \alpha_1 \\
0 & 1 & \alpha_4 & 0 & \alpha_2 & \alpha_2 \alpha_6 \\
\alpha_8 & 0 & 0 & 1 & \alpha_3 & \alpha_3 \alpha_6
\end{pmatrix}
\]
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\[
f_{10} \equiv \oint_{z = 0} \Omega_{20} = \int \frac{d\alpha_1}{\alpha_1} \wedge \ldots \wedge \frac{d\alpha_8}{\alpha_8} \delta^{3 \times 4} (C(\alpha) \cdot \tilde{\eta}) \delta^{3 \times 2} (C(\alpha) \cdot \tilde{\lambda}) \delta^{2 \times 3} (\lambda \cdot C^\perp (\alpha))
\]

\[
C(\alpha) \equiv \begin{pmatrix}
\alpha_6 & \alpha_8 & \alpha_1 & 1 & \alpha_6 & \alpha_1 & \alpha_7 & 0 \\
\alpha_8 & 0 & 0 & 1 & \alpha_5 & \alpha_4 \\
\alpha_3 & \alpha_2 & 0 & 0 & \alpha_2 & \alpha_7 & 1
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$$C(\alpha) \equiv \begin{pmatrix} \alpha_6 \alpha_8 & \alpha_1 & 1 & \alpha_6 & \alpha_1 & \alpha_7 & 0 \\ \alpha_8 & 0 & 0 & 1 & \alpha_5 & \alpha_4 \\ \alpha_3 & \alpha_2 & 0 & 0 & \alpha_2 & \alpha_7 & 1 \end{pmatrix}$$
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