Higher permutohedra at one-loop

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June 12, 2018
Objectives: (1) sketch a new interpretation of Parke-Taylor factors using an algebra of permutohedral tilings $\mathcal{V}^n$, and (2) state basic but essential results about the interpretation.

**Theorem[E]:** $\mathcal{V}^n$ is generated using *Minkowski sums of embedded trivalent graphs*. Any triangulation of an $n$-gon with a given cyclic vertex ordering defines a product of trivalent graphs; this product is independent of the triangulation!

A paper describing the combinatorial framework is in preparation.

Some back-history: Ph.D. thesis on symmetric group representations of generalized permutohedra; 1712.08520 (detailed study of characteristic functions of permutohedral cones, as *plates*), 1804.05460 (generalized permutohedra in the kinematic space).
• **Constructing the bridge:** define $x_{ij} = x_i - x_j$, set $\sigma = (1, 2, \ldots, n)$; put

$$PT(\sigma)(x) = \frac{1}{x_{12}x_{23} \cdots x_{n1}}.$$

• Setting $x_i = e^{-\varepsilon y_i}$, where $\varepsilon$ is regarded as a formal dilation parameter, after the naive transformation $y_i \mapsto y_i - \frac{1}{n} \sum_{j=1}^{n} y_j$ we have...

$$PT(\sigma)(x) \mapsto \frac{x_1 \cdots x_n}{x_{12}x_{23} \cdots x_{n1}} = \frac{e^{-\varepsilon (y_1 + \cdots + y_n)}}{e^{-\varepsilon y_{12}} e^{-\varepsilon y_{23}} \cdots e^{-\varepsilon y_{n1}}}.$$ 

• Taking the first two nonzero terms in the series expansion in $\varepsilon$ we get:

$$PT(\sigma)(y)\varepsilon^{-n} + \frac{1}{12} PT_L(\sigma)(y)\varepsilon^{-(n-2)} + \cdots ,$$

where the coefficient of $\varepsilon^{-(n-2)}$ is the elementary symmetric function

$$PT_L(\sigma)(y) = \sum_{1 \leq i < j \leq n} \frac{1}{y_{12} \cdots \widehat{y_{i,i+1}} \cdots \widehat{y_{j,j+1}} \cdots y_{n1}}.$$
We now interpret $\mathcal{PT}_L(\sigma)$ in terms of an algebra $\mathcal{V}^n$ of permutohedra...

Denote $u_{ij} = y_{ij}^{-1}$ and set(!) $v_{ijk} = u_{ij} + u_{jk} + u_{ki}$.

These satisfy some relations:

Antisymmetry: $u_{ij} = -u_{ji}$ and $v_{ijk} = v_{jki} = -v_{ikj}$

Linear straightening: $v_{ijk} - v_{jk\ell} + v_{k\ell i} - v_{\ell ij} = 0$

"Jacobi:" $u_{ij}u_{jk} + u_{jk}u_{ki} + u_{ki}u_{ij} = 0$, $v_{ijk}v_{ik\ell} + v_{ik\ell}v_{i\ell j} + v_{i\ell j}v_{ijk} = 0$.

Proposition[E]: modulo the ideal generated by $u_{ij}^2$ we have the square move $v_{124} v_{234} = v_{123} v_{134}$. 

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Theorem[E]. To each triangulation $\mathcal{I} = \{(i_1, j_2, k_1), \ldots, (i_{n-2}, j_{n-2}, k_{n-2})\}$, $(i_a < j_a < k_a)$ of an $n$-gon with cyclically ordered vertices $(1, \ldots, n)$, define

$$((\mathcal{I})) := v_{i_1j_1k_1} \cdots v_{i_{n-2}j_{n-2}k_{n-2}}.$$ 

Then, (1) modulo the ideal generated by the $u_{ij}^2$'s, $((\mathcal{I}))$ is independent of the triangulation, and in fact $((\mathcal{I})) = \text{PT}_L(1, 2, \ldots, n)$. (2) There is a canonical basis for $\mathcal{V}^n$ with graded dimension the Stirling numbers of the first kind.

Example:

$$v_{123}v_{134} = (u_{12} + u_{23} + u_{31})(u_{13} + u_{34} + u_{41})$$

$$= u_{12}u_{23} + u_{12}u_{34} + u_{12}u_{41} + u_{23}u_{34} + u_{23}u_{41} + u_{34}u_{41}$$

Sketch of Proof of (1). Use $\exp(u_{ij}) = 1 + u_{ij}$ and $\exp(v_{ijk}) = 1 + v_{ijk}$ modulo $u_{ij}^2$'s. Then, edges cancel additively!
Thank you!

Polytope for the Parke-Taylor factor PT(1, 2, 3, 4) (green), cut through by the six sheets for $\text{PT}_L(1, 2, 3, 4)$ (white).
Early, N. “Generalized permutohedra in the kinematic space.” 1804.05460.
Early, N. “Canonical Bases for Permutohedral Plates.” 1712.08520.
Early, N. “Permutohedral Blades.” In preparation.