Amplituhedron meets Jeffrey-Kirwan Residue

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with Livia Ferro and Tomasz Łukowski, arXiv:1805.01301
The Amplituhedron $\mathcal{A}$ - a recently discovered mathematical object:

- is a generalization of polytopes inside the Grassmannian
- has elements of the form $Y = C \cdot Z$
- has Positivity: $C, Z$ have all ordered maximal minors positive
- is equipped with a volume function $\Omega$ such that $\Omega$ has logarithmic singularities at all boundaries of $\mathcal{A}$

Tree Amplitudes in $\mathcal{N} = 4$ SYM can be extracted from $\Omega$

Geometrically

Triangulate $\mathcal{A}$ & Sum over volumes of triangles

$$\Omega = [123] + [134] + [145]$$

Analytically $\leftarrow$

Evaluate a contour integral

$$\Omega = \int_{\gamma} \omega$$

different triangulations $\leftrightarrow$ different contours
The **Jeffrey-Kirwan Residue** is an operation on Differential Forms [Jeffrey, Kirwan, ’95]

$$\omega = \frac{dx_1 \wedge \ldots \wedge dx_r}{\beta_1(x) \ldots \beta_n(x)}, \quad \beta_i(x) = \beta_i \cdot x + \alpha_i$$

- For $B = \{\beta_i\}$ and fixed $\eta \in \mathbb{R}^r$, it is defined as

$$\text{JKRes}^{B,\eta}_\omega = \sum_{\text{Cone} \ni \eta} \text{Res}_{\text{Cone}} \omega$$

- **Remarkable Property**

JKRes is *independent* from the chamber

- e.g. $\text{JKRes}^{B,\eta_1} = \text{Res}_{C_{25}} + \text{Res}_{C_{45}} + \text{Res}_{C_{23}}$
- $\parallel$
  - $\text{JKRes}^{B,\eta_2} = \text{Res}_{C_{45}} + \text{Res}_{C_{12}} + \text{Res}_{C_{42}}$
For Cyclic Polytopes and Conjugates (not Polytopes!):

\[ \Omega = \text{JKRes}^{B,\eta}_\omega \]  

[Ferro, Lukowski, MP, '18]

- Positivity of Amplituhedron \( \leftrightarrow \) configuration of Chambers

\rightarrow Geometrically

- each Chamber

triangulation of \( \mathcal{A} \)

representaion of \( \Omega \)

\[ \text{JKRes}^{B,\eta_1}_\omega = [134] + [123] + [145] \]

\[ \|

\[ \text{JKRes}^{B,\eta_2}_\omega = [345] + [351] + [312] \]

- adjacent Chambers

\[ \begin{array}{c}
\begin{array}{c}
2 \\
1 \\
3 \\
4 \\
5 \\
\end{array}
\end{array} = \begin{array}{c}
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Bistellar Flip

Global Residue Theorem
Amplituhedron meets Jeffrey-Kirwan Residue

For **Cyclic Polytopes** and **Conjugates** (*not Polytopes!*):

\[ \Omega = \text{JKRes}^{B,\eta}_\omega \]  

[Ferro, Lukowski, MP, ’18]

- **Positivity** of Amplituhedron \(\leftrightarrow\) **configuration** of **Chambers**
  
  \[ \rightarrow \text{Geometrically} \]
  \[ \text{Analytically} \leftarrow \]

- each Chamber

  triangulation of \(A\)

  \[ \text{JKRes}^{B,\eta_1}_\omega = [134] + [123] + [145] \]

  \[ \text{JKRes}^{B,\eta_2}_\omega = [345] + [351] + [312] \]

- adjacent Chambers

\[ \boxed{=} \]

Bistellar Flip

\[ [134] + [145] = [135] + [345] \]

Global Residue Theorem

**Secondary Polytope** \(\Sigma(\mathcal{P})\):

vertices are triangulations of \(\mathcal{P}\)

e.g. \(\Sigma(\text{n-gon}) = \text{Associahedron}\)

\[ \mathcal{P} \rightarrow A \]

Secondary Amplituhedron
The Jeffrey-Kirwan Residue

- computes the volume of polytopes and their parity conjugates (not polytopes!)
- encodes all triangulations, in a triangulation-independent way
- points at the Secondary Amplituhedron, generalising Secondary Polytopes

Open Questions

- What is the generalisation of the Jeffrey-Kirwan Residue for all other Amplituhedra, i.e. other helicity sectors and loops?
- Can we find the Secondary Amplituhedron in all these cases?
  → many new representations of Scattering Amplitudes!
\[ \Sigma(\text{Hexagon}) = \text{Pentagon} \]

Thank you!