The Physical Vernacular of the S-Matrix
On-Shell, All-Order Recursion Relations for Scattering Amplitudes

The Vernacular of the S-Matrix

Jacob L. Bourjaily

Amplitudes 2018 Summer School
QMAP, University of California, Davis

The Niels Bohr International Academy
The Vernacular of the S-Matrix

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Enormous Advances in Understanding Scattering Amplitudes
Enormous Advances in Understanding Scattering Amplitudes

\[ A_n = \sum_{L,R} (\text{diagram 1}) + (\text{diagram 2}) \]
Enormous Advances in Understanding Scattering Amplitudes

\[
A_n = \sum_{L,R} L \cdot R + I
\]

Part I: The Vernacular of the S-Matrix
Enormous Advances in Understanding Scattering Amplitudes

$A_n = \sum_{L,R} L \cdot R + \text{higher-order diagrams}$
The Physical Vernacular of the S-Matrix
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Enormous Advances in Understanding Scattering Amplitudes

\[ A_n = \sum_{L,R} L + R \]

\[ y^* \quad x^* \]

\[ i \]

\[ j \]

\[ i \]
Enormous Advances in Understanding Scattering Amplitudes

\[ A_n = \sum_{L,R} L \cdot R \]
Enormous Advances in Understanding Scattering Amplitudes

\[ A_n = \sum_{L,R} \left( \begin{array}{c} L \end{array} \right) \left( \begin{array}{c} R \end{array} \right) + \left( \begin{array}{c} \cdot \cdot \cdot \end{array} \right) \]
Enormous Advances in Understanding Scattering Amplitudes
On-Shell Physics/Grassmannian Geometry Correspondence
On-Shell Physics/Grassmannian Geometry Corresponcence

Important Open Questions

• how many functions exist? (how to name them?)
• what (functional) relations do they satisfy?
• what are their (infinite-dimensional) symmetries? – do these extend to entire amplitudes?
• do loop-level recursion relations exist?

On-Shell Physics

• on-shell diagrams – bi-colored, un-directed, planar
• physical symmetries – trivial symmetries (identities)

Grassmannian Geometry

• \{ \text{strata } C \in \text{Grassmanian}_k^n \}, \text{volume-form } \Omega_C
• \text{cluster variety } ( \prod_i d\alpha_i \alpha_i ) \times J_{N-4}
• \text{positroid variety } ( \prod_i d\alpha_i \alpha_i ) \times J_{N-4}
• \text{cluster variety } ( \text{?} ) \times J_{N-4}
• \text{volume-preserving diffeomorphisms } – \text{cluster coordinate mutations } C_\perp \equiv ( ) \Omega_C \equiv ( d\alpha_1 \alpha_1 \wedge \cdots \wedge d\alpha_{14} \alpha_{14} )
On-Shell Physics/Grassmannian Geometry Correspondence

\[ f_\Gamma \equiv \prod_i \left( \sum_{h_i, q_i} \int d^3 LIPS_i \right) \prod_v A_v \]
On-Shell Physics/Grassmannian Geometry Correspondence

\[ f_\Gamma \equiv \prod_i \left( \sum_{h_i, q_i} \int d^3 \text{LIPS}_i \right) \prod_v A_v \equiv \int \Omega_C \delta(C, p, h) \]
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On-Shell Physics/Grassmannian Geometry Correspondence

\[ f_\Gamma \equiv \prod_i \left( \sum_{h_i, q_i} \int d^3 \text{LIPS}_i \right) \prod_v A_v \equiv \int \Omega_C \delta(C, p, h) \]
On-Shell Physics/Grassmannian Geometry Correspondence

\[ f_\Gamma \equiv \prod_i \left( \sum_{h_i, q_i} \int d^3 L \right) \prod A_v \equiv \int \Omega_C \delta(C, p, h) \]
\[ f_\Gamma \equiv \prod_i \left( \sum_{h_i, q_i} \int d^3 \text{LIPS}_i \right) \prod_v A_v \equiv \int \Omega_C \delta(C, p, h) \]

On-Shell Physics/Grassmannian Geometry Correspondence

On-Shell Physics

Grassmannian Geometry

\[ \Omega_C \equiv \left( \int \text{LIPS}_i \right) \prod_v A_v \equiv \int \Omega_C \delta(C, p, h) \]
On-Shell Physics/Grassmannian Geometry Correspondence

\[ f_\Gamma \equiv \prod_i \left( \sum_{h_i,q_i} \int d^3 \text{LIPS}_i \right) \prod_v A_v \equiv \int \Omega_C \delta(C, p, h) \]

**On-Shell Physics**
- on-shell diagrams

**Grassmannian Geometry**
On-Shell Physics/Grassmannian Geometry Correspondence

\[ f_\Gamma \equiv \prod_i \left( \sum_{h_i, q_i} \int d^3 \text{LIPS}_i \right) \prod_v A_v \equiv \int \Omega_C \delta(C, p, h) \]

**On-Shell Physics**
- on-shell diagrams

**Grassmannian Geometry**
- \{strata \( C \in G(k, n) \), volume-form \( \Omega_C \)\}

![Graphical representation of on-shell diagrams and Grassmannian geometry]

**Important Open Questions**
- how many functions exist? (how to name them?)
- what (functional) relations do they satisfy?
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  - do these extend to entire amplitudes?
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**On-Shell Physics**
- on-shell diagrams

**Grassmannian Geometry**
- \{strata \( C \in G(k, n) \), volume-form \( \Omega_C \)\}
On-Shell Physics/Grassmannian Geometry Correspondence

\[ f_\Gamma \equiv \prod_i \left( \sum_{h_i, q_i} \int d^3 \text{LIPS}_i \right) \prod_v A_v \equiv \int \Omega_c \delta(C, p, h) \]

**On-Shell Physics**
- on-shell diagrams

**Grassmannian Geometry**
- \{strata \( C \in \text{G}(k, n) \), volume-form \( \Omega_c \)\}

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On-Shell Physics/Grassmannian Geometry Correspondence

\[ f_\Gamma \equiv \prod_i \left( \sum_{h_i, q_i} \int d^3LIPS_i \right) \prod_v A_v \equiv \int \Omega_C \delta(C, p, h) \]

**On-Shell Physics**
- on-shell diagrams

**Grassmannian Geometry**
- \{strata \( C \in G(k, n) \), volume-form \( \Omega_C \)\}
- volume-preserving diffeomorphisms
On-Shell Physics/Grassmannian Geometry Correspondence

\[ f_\Gamma \equiv \prod_i \left( \sum_{h_i, q_i} \int d^3 LIPS_i \right) \prod_v A_v \equiv \int \Omega_C \delta(C, p, h) \]

On-Shell Physics
- on-shell diagrams
- physical symmetries

Grassmannian Geometry
- \(\{\text{strata } C \in G(k, n), \text{ volume-form } \Omega_C\}\)
- volume-preserving diffeomorphisms
On-Shell Physics/Grassmannian Geometry Correspondence

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**On-Shell Physics**
- on-shell diagrams
- physical symmetries
  - trivial symmetries (identities)

**Grassmannian Geometry**
- \{strata \( C \in G(k, n) \), volume-form \( \Omega_C \)\}
- volume-preserving diffeomorphisms

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On-Shell Physics
- on-shell diagrams
- physical symmetries
  - trivial symmetries (identities)

Grassmannian Geometry
- \{strata \( C \in G(k, n) \), volume-form \( \Omega_C \)\}
- volume-preserving diffeomorphisms
  - cluster coordinate mutations

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\[ f_\Gamma \equiv \prod_i \left( \sum_{h_i, q_i} \int d^3 \text{LIPS}_i \right) \prod_v A_v \equiv \int \Omega_C \, \delta(C, p, h) \]

On-Shell Physics: planar $\mathcal{N} = 4$

- on-shell diagrams
- physical symmetries
  - trivial symmetries (identities)

Grassmannian Geometry

- \{strata $C \in G(k, n)$, volume-form $\Omega_C\}$
- volume-preserving diffeomorphisms
  - cluster coordinate mutations
On-Shell Physics/Grassmannian Geometry Correspondence

\[ f_\Gamma \equiv \prod_i \left( \sum_{h_i, q_i} \int d^3 \text{LIPS}_i \right) \prod_v A_v \equiv \int \Omega_C \delta(C, p, h) \]

On-Shell Physics: planar \( \mathcal{N} = 4 \)
- on-shell diagrams
- physical symmetries
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Grassmannian Geometry
- \{strata \( C \in G(k, n)\), volume-form \( \Omega_C \}\)
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On-Shell Physics: planar $\mathcal{N} = 4$

- on-shell diagrams
  - bi-colored
- physical symmetries
  - trivial symmetries (identities)

Grassmannian Geometry

- \{strata $C \in G(k, n)$, volume-form $\Omega_C$\}
- volume-preserving diffeomorphisms
  - cluster coordinate mutations
On-Shell Physics/Grassmannian Geometry Correspondence

\[ f_\Gamma \equiv \prod_i \left( \sum_{h_i q_i} \int d^3 \text{LIPS}_i \right) \prod_v A_v \equiv \int \Omega_C \delta(C, p, h) \]

On-Shell Physics: planar \( \mathcal{N} = 4 \)
- on-shell diagrams
  - bi-colored, undirected
- physical symmetries
  - trivial symmetries (identities)

Grassmannian Geometry
- \( \{\text{strata } C \in G(k, n), \text{ volume-form } \Omega_C\} \)
- volume-preserving diffeomorphisms
  - cluster coordinate mutations
On-Shell Physics/Grassmannian Geometry Correspondence

\[ f_\Gamma \equiv \prod_i \left( \sum_{h_i, q_i} \int d^3 \text{LIPS}_i \right) \prod_v A_v \equiv \int \Omega_C \delta(C, p, h) \]

On-Shell Physics: \( \text{planar } \mathcal{N}=4 \)
- on-shell diagrams
  - bi-colored, \textbf{undirected}, \textbf{planar}
- physical symmetries
  - trivial symmetries (identities)

Grassmannian Geometry
- \( \{\text{strata } C \in G(k, n), \text{ volume-form } \Omega_C\} \)
- volume-preserving diffeomorphisms
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On-Shell Physics/Grassmannian Geometry Correspondence

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f_\Gamma \equiv \prod_i \left( \sum_{h_i, q_i} \int d^3 \text{LIPS}_i \right) \prod_v A_v \equiv \int \Omega_C \delta(C, p, h)
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On-Shell Physics: planar $\mathcal{N}=4$
- on-shell diagrams
  - bi-colored, undirected, planar
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Grassmannian Geometry
- \{strata $C \in G(k, n)$, volume-form $\Omega_C$\}
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**On-Shell Physics:** planar \( \mathcal{N} = 4 \)
- on-shell diagrams
  - bi-colored, undirected, planar
- physical symmetries
  - trivial symmetries (identities)

**Grassmannian Geometry**
- \( \{ \text{strata } C \in G(k, n), \text{volume-form } \Omega_C \} \)
- volume-preserving diffeomorphisms
  - cluster coordinate mutations

\[ C \equiv \begin{pmatrix}
1 & \alpha_8 & \alpha_8 + \alpha_{14} & \alpha_5 \alpha_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \alpha_{10} \alpha_4 + \alpha_{10} \alpha_{13} & \alpha_4 \alpha_7 & 0 & 0 \\
\alpha_3 \alpha_9 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 \alpha_3 + \alpha_6 \alpha_{12} \\
\alpha_9 & 0 & \alpha_1 & \alpha_1 \alpha_{11} & 0 & \alpha_1 \alpha_2 & \alpha_1 \alpha_2 \alpha_7 & 0 \\
\end{pmatrix} \]

\[ \Omega_C \equiv \left( \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_{14}}{\alpha_{14}} \right) \]
On-Shell Physics/Grassmannian Geometry Correspondence

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On-Shell Physics: planar $\mathcal{N} = 4$
- on-shell diagrams
  - bi-colored, undirected, planar
- physical symmetries
  - trivial symmetries (identities)

Grassmannian Geometry
- \{strata $C \in G(k, n)$, volume-form $\Omega_C$\}
  - positroid variety
- volume-preserving diffeomorphisms
  - cluster coordinate mutations

\[ C \equiv \begin{pmatrix} 1 & \alpha_8 & \alpha_5 + \alpha_8 \alpha_{14} & \alpha_5 \alpha_{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha_{10} \alpha_4 + \alpha_{10} \alpha_{13} & \alpha_{13} \alpha_7 & 0 & 0 \\ \alpha_3 \alpha_9 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 \alpha_3 + \alpha_6 \alpha_{12} \end{pmatrix} \]

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- on-shell diagrams
  - bi-colored, undirected, planar
- physical symmetries
  - trivial symmetries (identities)

Grassmannian Geometry

- \{strata \( C \in G(k, n) \), volume-form \( \Omega_C \}\)
  - positroid variety, \( (\prod_i \frac{d\alpha_i}{\alpha_i}) \)
- volume-preserving diffeomorphisms
  - cluster coordinate mutations

\[ C \equiv \begin{pmatrix}
1 & \alpha_8 & \alpha_{8+14} & \alpha_{5+11} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \alpha_{10+4} & \alpha_{10+13} & \alpha_{4+7} & 0 & 0 \\
\alpha_3 \alpha_9 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & \alpha_{3+6\alpha_{12}} \\
\alpha_9 & \alpha_1 & \alpha_{1+11} & 0 & \alpha_1 \alpha_2 & \alpha_1 \alpha_2 \alpha_7 & 0 & 1
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On-Shell Physics: planar \( \mathcal{N} = 4 \)
- on-shell diagrams
  - bi-colored, un\textbf{d}irected, \textbf{p}lanar
- physical symmetries
  - trivial symmetries (identities)

Grassmannian Geometry
- \{strata \( C \in G(k, n) \), volume-form \( \Omega_C \}\)
  - positroid variety \( \left( \prod_i \frac{d\alpha_i}{\alpha_i} \right) \)
- volume-preserving diffeomorphisms
  - cluster coordinate mutations

\[ C \equiv \begin{pmatrix}
1 & \alpha_8 & \alpha_8 + \alpha_{14} & \alpha_5 \alpha_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \alpha_{10} \alpha_{4} + \alpha_{10} \alpha_{13} & \alpha_4 \alpha_7 & 0 & 0 \\
\alpha_3 \alpha_9 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 \alpha_3 + \alpha_6 \alpha_{12} \\
\alpha_9 & 0 & \alpha_1 & \alpha_1 \alpha_{11} & 0 & \alpha_1 \alpha_2 & \alpha_1 \alpha_2 \alpha_7 & 0 \\
\alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 & 1 & 0
\end{pmatrix} \]

\[ \Omega_C \equiv \left( \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_{14}}{\alpha_{14}} \right) \]
On-Shell Physics/Grassmannian Geometry Correspondence

\[ f_\Gamma \equiv \prod_i \left( \sum_{h_i, q_i} \int d^3 LIP_{S_i} \right) \prod_v A_v \equiv \int \Omega_C \, \delta(C, p, h) \]

On-Shell Physics: planar $\mathcal{N} = 4$

- on-shell diagrams
  - bi-colored, **undirected**, **planar**
- physical symmetries
  - trivial symmetries (identities)

Grassmannian Geometry

- \{strata $C \in G(k, n)$, volume-form $\Omega_C$\}
  - positroid variety, \( \prod_i \frac{d\alpha_i}{\alpha_i} \)
- volume-preserving diffeomorphisms
  - cluster coordinate mutations

\[ C \equiv \begin{pmatrix}
1 & \alpha_8 & \alpha_5 + \alpha_8 \alpha_{14} & \alpha_5 \alpha_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \alpha_{10} \alpha_4 + \alpha_{10} \alpha_{13} & \alpha_4 \alpha_7 & 0 & 0 \\
\alpha_3 \alpha_9 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 \alpha_3 + \alpha_6 \alpha_{12} \\
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\end{pmatrix} \]

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On-Shell Physics: planar \( \mathcal{N} = 4 \)

- on-shell diagrams
  - bi-colored, undirected, **planar**
- physical symmetries: the **Yangian**
  - trivial symmetries (identities)

Grassmannian Geometry

- \{strata \( C \in G(k, n) \), volume-form \( \Omega_C \}\)
  - positroid variety, \( (\prod_i \frac{d\alpha_i}{\alpha_i}) \)
- volume-preserving diffeomorphisms
  - cluster coordinate mutations

\[ C \equiv \begin{pmatrix}
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\alpha_3 \alpha_9 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 \alpha_3 + \alpha_6 \alpha_{12} \\
\alpha_9 & 0 & \alpha_1 & \alpha_1 \alpha_{11} & 0 & \alpha_1 \alpha_2 & \alpha_1 \alpha_2 \alpha_7 & 0 \\
\end{pmatrix} \]

\[ \Omega_C \equiv \left( \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_{14}}{\alpha_{14}} \right) \]
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On-Shell Physics: planar $\mathcal{N} = 4$
- on-shell diagrams
  - bi-colored, undirected, planar
- physical symmetries: the Yangian
  - trivial symmetries (identities)

Grassmannian Geometry
- \{strata $C \in G(k, n)$, volume-form $\Omega_C$\}
  - positroid variety, \( \left( \prod_i \frac{d\alpha_i}{\alpha_i} \right) \)
- volume-preserving diffeomorphisms
  - cluster coordinate mutations

\[ \Omega_C \equiv \left( \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_{14}}{\alpha_{14}} \right) \]
On-Shell Physics/Grassmannian Geometry Correspondence

\[ f_\Gamma \equiv \prod_i \left( \sum_{h_i, q_i} \int d^3 \text{LIPS}_i \right) \prod_v A_v \equiv \int \Omega_C \delta(C, p, h) \]

**On-Shell Physics:** planar $\mathcal{N} < 4$

- on-shell diagrams
  - bi-colored, **directed**, planar
- physical symmetries: ?
  - trivial symmetries (identities)

**Grassmannian Geometry**

- \{strata $C \in G(k, n)$, volume-form $\Omega_C\}$
  - positroid variety, \( (\prod_i \frac{d\alpha_i}{\alpha_i}) \times \mathcal{J}^{\mathcal{N}-4} \)
- volume-preserving diffeomorphisms
  - cluster coordinate mutations

\[ C \equiv \begin{pmatrix}
1 & \alpha_8 & \alpha_{5}+\alpha_{8} & \alpha_{14} & \alpha_5\alpha_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \alpha_{10} & \alpha_{4}+\alpha_{10}\alpha_{13} & \alpha_{4}\alpha_{7} & 0 & 0 \\
\alpha_{3}\alpha_{9} & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_{6} & \alpha_{3}+\alpha_{6}\alpha_{12} \\
\alpha_{9} & 0 & \alpha_1 & \alpha_{1}\alpha_{11} & 0 & \alpha_{1}\alpha_{2} & \alpha_{1}\alpha_{2}\alpha_{7} & 0 & 1
\end{pmatrix} \]

\[ \Omega_C \equiv \left( \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_{14}}{\alpha_{14}} \right) \times (1)^{\mathcal{N}-4} \]
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On-Shell Physics: planar \( \mathcal{N} < 4 \)

- on-shell diagrams
  - bi-colored, directed, planar
- physical symmetries: ?
  - trivial symmetries (identities)

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- \( \{ \text{strata } C \in G(k, n), \text{ volume-form } \Omega_C \} \)
  - positroid variety, \( \left( \prod_i \frac{d\alpha_i}{\alpha_i} \right) \times \mathcal{J}^{\mathcal{N}-4} \)
- volume-preserving diffeomorphisms
  - cluster coordinate mutations

\[ C \equiv \begin{pmatrix}
1 & \alpha_8 & \alpha_5 & \alpha_{14} & \alpha_5 & \alpha_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_{10} & \alpha_4 & \alpha_{10} & \alpha_{13} & \alpha_4 & \alpha_7 & 0 & 0 \\
\alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & \alpha_3 & \alpha_6 & \alpha_{12} & 0 & 0 & 0 & 0 \\
\alpha_9 & 0 & \alpha_1 & \alpha_1 & \alpha_{11} & 0 & \alpha_1 & \alpha_2 & \alpha_1 & \alpha_2 & \alpha_7 & 0 & 1 \end{pmatrix} \]

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\alpha_3 & \alpha_9 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & \alpha_3 + \alpha_6 & \alpha_{12} \\
\alpha_9 & 0 & \alpha_1 & \alpha_1 & \alpha_{11} & 0 & \alpha_1 & \alpha_2 & \alpha_1 & \alpha_2 & \alpha_7 & 0 & 1
\end{pmatrix} \]

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  - cluster variety
- volume-preserving diffeomorphisms
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![Diagram](image)

\[ C_\perp \equiv \begin{pmatrix} \alpha_1 & 1 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & 1 & 0 & \alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_5 & 1 & \alpha_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_7 & \alpha_8 & 0 & 0 & 0 & 0 \\ \alpha_11 & 0 & 0 & 0 & 0 & 0 & \alpha_9 & 1 & \alpha_{10} & 0 & 0 & 0 \\ 0 & \alpha_{13} & \alpha_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{12} & 0 & 0 \end{pmatrix} \]

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0 & 0 & 0 & \alpha_5 & 1 & \alpha_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_7 & 1 & \alpha_8 & 0 & 0 & 0 & 0 \\
\alpha_{11} & 0 & 0 & 0 & 0 & 0 & \alpha_9 & 1 & \alpha_{10} & 0 & 0 & 0 \\
0 & \alpha_{13} & \alpha_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_{12} & 0 \\
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On-Shell Physics: non-planar $\mathcal{N}<4$
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\[ C^\perp \equiv \left( \begin{array}{cccccccccc}
\alpha_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_5 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & \alpha_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_{13} & 0 & \alpha_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
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Important Open Questions

• how many functions exist? (how to name them?)
• what (functional) relations do they satisfy?
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Amplitudes 2018 Summer School  QMAP, University of California, Davis

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1. *Spiritus Movens*: a Parable from Collider Physics
   - A Simple, Practical Problem in Quantum Chromodynamics

2. The Physical Vernacular of the S-Matrix
   - Beyond (Mere) Scattering Amplitudes: On-Shell Functions
   - Physically Observable Data Describing Massless Particles in 4d
   - Basic Building Blocks: S-Matrices for Three Massless Particles

3. On-Shell, All-Order Recursion Relations for Scattering Amplitudes
   - Deriving Diagrammatic Recursion Relations for Amplitudes
   - *Exempli Gratia*: On-Shell Representations of Tree Amplitudes
   - On-Shell Representations of Loop-Amplitude Integrands
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BY GLUON–GLUON FUSION

Stephen J. PARKE and T.R. TAYLOR

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*Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510 USA*

Received 13 September 1985

The cross section for two-gluon to four-gluon scattering is given in a form suitable for fast numerical calculations.
Supercomputer Computations in Quantum Chromodynamics

Consider the amplitude for two gluons to collide and produce four: \( gg \rightarrow gggg \). Before modern computers, this would have been computationally intractable

- 220 Feynman diagrams, thousands of terms
- In 1985, Parke and Taylor took up the challenge
  - using every theoretical tool available
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\[
\begin{align*}
D_1^{(24)} &= -2 \frac{s_{34}^2}{t_{24} t_{13}} E(p_1, p_3) E(p_2, p_4), \\
D_1^{(25)} &= -2 \frac{t_{24}}{s_{12} s_{13}} E(p_1, p_2) E(p_3, p_4), \\
D_1^{(26)} &= -2 \frac{s_{12} s_{13}}{t_{24}} E(p_1, p_3) E(p_2, p_4), \\
D_1^{(27)} &= -2 \frac{s_{34}}{s_{12} s_{13}} E(p_1, p_4) E(p_2, p_3), \\
D_1^{(28)} &= -2 \frac{t_{24}}{s_{12} s_{13}} E(p_1, p_2) E(p_3, p_4), \\
D_1^{(29)} &= -2 \frac{s_{34}}{s_{12} s_{13}} E(p_1, p_4) E(p_2, p_3), \\
D_1^{(30)} &= -\frac{4}{s_{12} s_{13}} [(p_1 + p_2 - p_3)(p_4 + p_3 - p_4) - t_{12}] E(p_1, p_2), \\
D_1^{(31)} &= -\frac{4}{s_{12} s_{13}} [(p_1 + p_2 - p_3)(p_4 + p_3 - p_4) + t_{12}] E(p_1, p_2), \\
D_1^{(32)} &= -\frac{4}{s_{12} s_{13}} [(p_1 - p_2 + p_4)(p_3 + p_4 - p_1) + t_{12}] E(p_1, p_2), \\
D_1^{(33)} &= -\frac{4}{s_{12} s_{13}} [(p_1 - p_2 + p_4)(p_3 + p_4 - p_1) - t_{12}] E(p_1, p_2).
\end{align*}
\]

where \( t_{12} = 1 \).

The diagrams \( D_1^{(i)} \) are obtained from \( D_1^{(i)} \) by replacing \( t_{12} \) by \( t_{12} = 0 \) and the functions

\( E(p_i, p_j) \) by \( G(p_i, p_j) \).

The diagrams \( D_2^{(j)} \) are listed below:

\[
\begin{align*}
D_2^{(1)} &= -\frac{4}{s_{12} s_{13}} (F(p_1, p_3) E(p_2, p_4) - F(p_2, p_3) E(p_1, p_4)) \\
&+ (F(p_1, p_3) + s_{12}) E(p_2, p_4), \\
D_2^{(2)} &= -\frac{4}{s_{12} s_{13}} (F(p_1, p_3) + s_{12}) E(p_2, p_4) \\
&+ F(p_1, p_3) E(p_2, p_4) - F(p_1, p_3) E(p_2, p_4), \\
D_2^{(3)} &= -\frac{4}{s_{12} s_{13}} - (F(p_1, p_3) - F(p_1, p_3) E(p_2, p_4)) \\
&- (F(p_1, p_3) E(p_2, p_4) - F(p_1, p_3) E(p_2, p_4)).
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[3, 4], convoluted with the appropriate Altarelli-Parisi probabilities [5]. Our result has successfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist’s, but also a theorist’s delight.

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The Discovery of Incredible, Unanticipated Simplicity

They soon guessed a simplified form of the amplitude

\[ \langle a b \rangle_{4} \langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \langle 5 6 \rangle \langle 6 1 \rangle \delta^{2} \times \delta^{2} (\lambda \cdot \tilde{\lambda}) \]

Here, we have used spinor variables to describe the external momenta:

\[ \tilde{\lambda}^\alpha_\beta \equiv \lambda^\alpha_\beta \]

Notice that

\[ p^\mu p^\mu = \det(\lambda^\alpha_\beta) = 0 \]

for massless particles.
The Discovery of Incredible, Unanticipated Simplicity

They soon guessed a simplified form of the amplitude (checked numerically):

\[
\langle a b \rangle_4 \langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \langle 5 6 \rangle \langle 6 1 \rangle \delta^2 \times \frac{1}{2} (\lambda \cdot \bar{\lambda})^2 (n \sum_{a=1}^{4} \lambda^a \bar{\lambda}^a) \]

Here, we have used spinor variables to describe the external momenta:

\[
\tilde{\lambda}^a \bar{\lambda}^a \rightarrow p^a \mu \equiv p^a_0 + ip^a_2 - ip^a_1 + ip^a_3.
\]

Notice that

\[
p^a \mu p^a \mu = \det (\lambda a, \bar{\lambda} a) = 0
\]

for massless particles.

This is made manifest by the (local) Lorentz group, \( SL(2) \times SL(2) \), acting on \( \lambda a \) and \( \tilde{\lambda} a \), respectively. The Grassmannian \( G(k, n) \): the linear span of \( k \) vectors in \( C^n \).

Momentum conservation becomes the geometric statement:

\( \lambda \subset \tilde{\lambda} \perp \) and \( \tilde{\lambda} \subset \lambda \perp \).

Thus, Lorentz invariants must be constructed out of determinants:

\[
\langle a b \rangle \equiv \det (\lambda a, \lambda b), \quad [a b] \equiv \det (\tilde{\lambda} a, \tilde{\lambda} b)
\]

The action of the little group corresponds to:

\[
(\lambda a, \tilde{\lambda} a) \rightarrow t a \lambda a, t^{-1} a \tilde{\lambda} a)
\]

Amplitudes 2018 Summer School  QMAP, University of California, Davis

Part I: The Vernacular of the S-Matrix
The Discovery of Incredible, Unanticipated Simplicity

They soon guessed a simplified form of the amplitude (checked numerically):

\[ \langle a b \rangle_4 \langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \langle 5 6 \rangle \langle 6 1 \rangle \delta_2^2 = \lambda \cdot \tilde{\lambda} \]

Here, we have used spinor variables to describe the external momenta:

\[ \tilde{\lambda}_a \dot{\lambda}_b = p_{\mu a} \sigma_{\alpha \dot{\alpha}} p^{\alpha \dot{\alpha}} \]

Notice that

\[ p_{\mu a} p^{\mu a} = \det (\lambda_a \dot{\lambda}_a) = 0 \]

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\langle a \ b \rangle^4 \over \langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \langle 3 \ 4 \rangle \langle 4 \ 5 \rangle \langle 5 \ 6 \rangle \langle 6 \ 1 \rangle \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda})
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\[
\begin{align*}
\langle a b \rangle^4 &= \frac{\langle a b \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle} \delta^2 (\lambda \cdot \tilde{\lambda})
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On What *Data* Does a Scattering Amplitude Depend?

A scattering amplitude, $A_n$, can be a generally complicated(?) function of all the *physically observable data* describing each of the particles involved.
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$$A_n \equiv \begin{array}{c}
\text{physical data for the } a\text{th particle:} \\
|a\rangle \cdot p_a \mu \\
\text{momentum, on-shell: } p_a^2 = 0 \\
\text{all the non-kinematical quantum numbers of } a\text{ (color, flavor, ...)} \\
\text{Although a Lagrangian formalism requires that we use polarization tensors, it is impossible to continuously define polarizations for each helicity state without introducing unobservable (gauge) redundancy — e.g. for } \sigma_a = 1: \\
\epsilon_{\mu a} \sim \epsilon_{\mu a} + \alpha(p_a) p_{\mu a}
\end{array}$$

Such unphysical baggage is almost certainly responsible for the incredible obfuscation of simplicity in the traditional approach to quantum field theory.
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$$\mathcal{A}_n \equiv \begin{array}{c}
\includegraphics[width=0.3\textwidth]{amplitude_diagram}
\end{array}$$

**Physical data for the $a^{th}$ particle: $|a\rangle$**

- $p_a^\mu$ momentum
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A general representation of the data for the $a^{th}$ particle is given by:

\[ A_n \equiv \begin{array}{c} 1 \\
\vdots \\
n \end{array} \]

Physical data for the $a^{th}$ particle: $|a\rangle$

- $p_a^\mu$ momentum, *on-shell*: $p_a^2 - m_a^2 = 0$
- $\sigma_a$ spin
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Physical data for the $a^{th}$ particle: $|a⟩$

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$\mathcal{A}_n \equiv \begin{array}{c} 1 \\ \cdots \cdots \\ n \end{array}$

Physical data for the $a^{th}$ particle: $|a\rangle$

- $p^\mu_a$ momentum, *on-shell*: $p^2_a - m^2_a = 0$
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\vcenter{\hbox{\includegraphics[width=0.5\textwidth]{figure}}}
\end{array}
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A scattering amplitude, \( A_n \), can be a generally complicated(?) function of all the *physically observable data* describing each of the particles involved.

\[
A_n \equiv \begin{array}{ccc}
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
n & & \\
\end{array}
\]

Although a Lagrangian formalism requires that we use polarization tensors, it is *impossible* to continuously define polarizations for each helicity state without introducing *unobservable* (gauge) redundancy—e.g. for \( \sigma_a = 1 \):

\[
\epsilon_a^{\mu} \sim \epsilon_a^{\mu} + \alpha(p_a)p_a^{\mu}
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Such *unphysical baggage* is almost certainly responsible for the incredible obfuscation of simplicity in the traditional approach to quantum field theory.
Broadening the Class of Physically Meaningful Functions

We are interested in the class of functions involving **only** observable quantities.
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**Internal Particles:**
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Internal Particles: locality dictates that we multiply each amplitude,
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\[ \mathcal{A}_L(\ldots, I) \times \mathcal{A}_R(I, \ldots) \]
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\int d^{d-1} \text{LIPS}_I \ A_L(\ldots, I) \times A_R(I, \ldots)
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\textbf{Internal Particles:} locality dictates that we multiply each amplitude, and unitarity dictates that we marginalize over unobserved states—integrating over the Lorentz-invariant phase space ("LIPS") for each particle $I$, and summing over the possible states

$$\int d^{d-1}\text{LIPS}_I \ A_L(\ldots, I) \times A_R(I, \ldots)$$
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\[
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**On-Shell Functions:**
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**On-Shell Functions**: networks of amplitudes, $A_v$, connected by any number of internal particles, $i \in I$, forming a graph $\Gamma$ called an “on-shell diagram”.
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\phi_\Gamma \equiv \prod_{i \in I} \left( \sum_{h_i, q_i, m_i, \ldots} \int d^{d-1} \text{LIPS}_i \right) \prod_v A_v
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\[ n_\delta \]
Broadening the Class of Physically Meaningful Functions

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**Counting Constraints**:

$$n_\delta \equiv d \times n_V - (d-1) \times n_I$$
Broadening the Class of Physically Meaningful Functions

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**On-Shell Functions**: networks of amplitudes, \( A_v \), connected by any number of internal particles, \( i \in I \), forming a graph \( \Gamma \) called an “on-shell diagram”.

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\[ \hat{n}_\delta \equiv d \times n_V - (d-1) \times n_I - d \]
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$$\hat{n}_\delta \equiv d \times n_V - (d-1) \times n_I - d = 0$$
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\[
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$$\hat{n}_\delta \equiv d \times n_V - (d-1) \times n_I - d < 0 \quad \Rightarrow \quad \text{(-} \hat{n}_\delta \text{) non-trivial integrations}$$
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Counting Constraints:

$$\hat{n}_\delta \equiv d \times n_V - (d-1) \times n_I - d = 0 \quad \Rightarrow \quad (\hat{n}_\delta) \text{ kinematical constraints}$$

$$\hat{n}_\delta < 0 \quad \Rightarrow \quad (\hat{n}_\delta) \text{ non-trivial integrations}$$

$$\hat{n}_\delta \equiv \hat{n}_{\text{kinematical}}$$

$$\hat{n}_\delta = \text{number of excess } \delta \text{-functions}$$

$$\hat{n}_\delta = \text{minus number of remaining integrations}$$
Making *Masslessness* Manifest: Spinor-Helicity Variables

To avoid *constraining* each particle’s momentum to be null, van der Waerden introduced (in 1929!) *spinor-helicity* variables to make this always trivial.
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\[ p^\mu_a \]

Notice that \( \det(\lambda_\alpha^a, \lambda_\beta^b) \equiv \langle a \ b \rangle \det(\tilde{\lambda}_\dot{\alpha}^a, \tilde{\lambda}_\dot{\beta}^b) \equiv \left[a \ b\right] \) for massless particles. This can be made manifest by writing *p\_a\_\alpha^\dot{\alpha}\_a* as an outer product of 2-vectors.

When \( p_a \) is real (\( p_a \in \mathbb{R}^3, 1 \)), \( p_\alpha^\dot{\alpha}_a = (p_\alpha^\dot{\alpha}_a)^\dagger \), which implies that \( (\lambda_\alpha^a)^* = \pm \tilde{\lambda}_\dot{\alpha}^a \).

(but allowing for complex momenta, \( \lambda_a \) and \( \tilde{\lambda}_a \) become independent.)

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\[ p^\mu_a \mapsto p^{\alpha\dot{\alpha}}_a \]

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\[ \equiv (p_0^a + ip_2^a)(p_1^a - ip_2^a)(p_3^a) \]

\[ \equiv \lambda^\alpha_a \tilde{\lambda}^{\dot{\alpha}}_a \]

\[ \langle a | h_a \mapsto t - 2h_a | a \rangle \]

\[ \text{Notice that } \det (p^{\alpha\dot{\alpha}}_a) = (p_0^a)^2(p_1^a)^2(p_2^a)^2(p_3^a)^2 = m_2^a, \text{ for massless particles.} \]

This can be made manifest by writing \( p^{\alpha\dot{\alpha}}_a \) as an outer product of 2-vectors.

When \( p_a \) is real \((p_a \in \mathbb{R}^3, 1)\), \( p^{\alpha\dot{\alpha}}_a = (p^{\alpha\dot{\alpha}}_a)^\dagger \), which implies that \( (\lambda^\alpha_a)^\ast = \pm \tilde{\lambda}^{\dot{\alpha}}_a \).

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\[ \det (\lambda^\alpha_a, \lambda^\beta_b) \equiv \langle a b \rangle \]

\[ \det (\tilde{\lambda}^{\dot{\alpha}}_a, \tilde{\lambda}^{\dot{\beta}}_b) \equiv [a b] \]

\[ \text{Amplitudes 2018 Summer School} \quad \text{QMAP, University of California, Davis} \]

Part I: *The Vernacular of the S-Matrix*
Making *Masslessness* Manifest: Spinor-Helicity Variables

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\[ p_a^\mu \mapsto p_a^{\alpha \dot{\alpha}} \equiv p_a^\mu \sigma_{\mu}^{\alpha \dot{\alpha}} \]
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\[
p^\mu_a \rightarrow p^\alpha_{\dot{\alpha}} \equiv p^\mu_a \sigma^\alpha_{\dot{\alpha}} = \begin{pmatrix} p^0_a + p^3_a & p^1_a - ip^2_a \\ p^1_a + ip^2_a & p^0_a - p^3_a \end{pmatrix}
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- Notice that $\det(p_a^{\alpha \dot{\alpha}}) = (p_a^0)^2 - (p_a^1)^2 - (p_a^2)^2 - (p_a^3)^2 = m_a^2$
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\[ p_\mu^a \mapsto p_\mu^\alpha \bar{\alpha} \equiv p_\mu^a \sigma_\mu^{\alpha \bar{\alpha}} = \left( \begin{array}{cc} p_0^a + p_3^a & p_1^a - ip_2^a \\ p_1^a + ip_2^a & p_0^a - p_3^a \end{array} \right) \equiv \lambda_\alpha^a \bar{\lambda}_{\bar{\alpha}}^a \]

- Notice that \( \det(p_\alpha^\bar{\alpha}) = (p_0^a)^2 - (p_1^a)^2 - (p_2^a)^2 - (p_3^a)^2 = 0 \), for massless particles. This can be made manifest by writing \( p_\alpha^\bar{\alpha} \) as an outer product of 2-vectors.

- When \( p_a \) is real \( (p_a \in \mathbb{R}^{3,1}) \), \( p_\alpha^\bar{\alpha} = (p_\alpha^\bar{\alpha})^\dagger \)
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\[
p^\mu_a \mapsto p^\alpha_\dot{\alpha} \equiv p^\mu_a \sigma^\alpha_{\mu} = \begin{pmatrix} p^0_a + p^3_a & p^1_a - i p^2_a \\ p^1_a + i p^2_a & p^0_a - p^3_a \end{pmatrix} \equiv \lambda^\alpha_a \tilde{\lambda}_{\dot{\alpha}}^a
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- Notice that \( \det(p^\alpha_\dot{\alpha}) = (p^0_a)^2 - (p^1_a)^2 - (p^2_a)^2 - (p^3_a)^2 = 0 \), for massless particles. This can be made manifest by writing \( p^\alpha_\dot{\alpha} \) as an outer product of 2-vectors.

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\[ p_a^\mu \mapsto p_a^{\alpha \dot{\alpha}} \equiv p_a^\mu \sigma_{\mu}^{\alpha \dot{\alpha}} = \begin{pmatrix} p_a^0 + p_a^3 & p_a^1 - ip_a^2 \\ p_a^1 + ip_a^2 & p_a^0 - p_a^3 \end{pmatrix} \equiv \lambda_a^{\alpha} \tilde{\lambda}_a^{\dot{\alpha}} \]

- Notice that \( \det(p_a^{\alpha \dot{\alpha}}) = (p_a^0)^2 - (p_a^1)^2 - (p_a^2)^2 - (p_a^3)^2 = 0 \), for massless particles. This can be made manifest by writing \( p_a^{\alpha \dot{\alpha}} \) as an outer product of 2-vectors.

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Thus, all the kinematical data can be described by a pair of \((2 \times n)\) matrices:
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\end{pmatrix}
\quad
\begin{pmatrix}
\tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \tilde{\lambda}_3^1 & \cdots & \tilde{\lambda}_n^1 \\
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\lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \cdots & \lambda_n^3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_1^n & \lambda_2^n & \lambda_3^n & \cdots & \lambda_n^n
\end{pmatrix}
\]

\[
\tilde{\lambda} \equiv \begin{pmatrix}
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\tilde{\lambda}_1^3 & \tilde{\lambda}_2^3 & \tilde{\lambda}_3^3 & \cdots & \tilde{\lambda}_n^3 \\
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\end{pmatrix} \equiv \begin{pmatrix}
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writing \(\lambda_a \in \mathbb{C}^2\) for a column, \(\lambda^\alpha \in \mathbb{C}^n\) for a row.
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- Because Lorentz transformations mix the rows of each matrix, \(\lambda^\alpha, \tilde{\lambda}^{\dot{\alpha}}\),

\[
\delta^2_4 (\sum_a p^\mu_a) = \delta^2_{2 \times 2} (\sum_a) \equiv \delta^2_{2 \times 2} (\lambda \cdot \tilde{\lambda})
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\lambda \equiv \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \end{pmatrix} \equiv \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} \quad \tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1 & \tilde{\lambda}_2 & \tilde{\lambda}_3 & \cdots & \tilde{\lambda}_n \end{pmatrix} \equiv \begin{pmatrix} \tilde{\lambda}^1 \\ \tilde{\lambda}^2 \end{pmatrix}
\]

writing \(\lambda_a \in \mathbb{C}^2\) for a column, \(\lambda^\alpha \in \mathbb{C}^n\) for a row.

- Because Lorentz transformations mix the rows of each matrix, \(\lambda^\alpha, \tilde{\lambda}^\dot{\alpha}\), and the little group allows for rescaling
The Grassmannian Geometry of Kinematical Constraints

Thus, all the kinematical data can be described by a pair of $(2 \times n)$ matrices:

\[
\lambda \equiv \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \end{pmatrix} \equiv \begin{pmatrix} \lambda_1^1 \\ \lambda_2^1 \end{pmatrix} \quad \tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1 & \tilde{\lambda}_2 & \tilde{\lambda}_3 & \cdots & \tilde{\lambda}_n \end{pmatrix} \equiv \begin{pmatrix} \tilde{\lambda}_1^1 \\ \tilde{\lambda}_2^1 \end{pmatrix}
\]

writing $\lambda_a \in \mathbb{C}^2$ for a column, $\lambda^{\alpha} \in \mathbb{C}^n$ for a row.

• Because Lorentz transformations mix the rows of each matrix, $\lambda^{\alpha}$, $\tilde{\lambda}^{\dot{\alpha}}$, and the little group allows for rescaling, the invariant content of the data is:
The Grassmannian Geometry of Kinematical Constraints

Thus, all the kinematical data can be described by a pair of $(2 \times n)$ matrices:

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- Because Lorentz transformations mix the rows of each matrix, $\lambda^\alpha$, $\tilde{\lambda}^{\dot{\alpha}}$, and the little group allows for rescaling, the invariant content of the data is:

  **The “two–plane” $\lambda$:**
  
  the span of 2 vectors $\lambda^\alpha \in \mathbb{C}^n$
The Grassmannian Geometry of Kinematical Constraints

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\lambda \equiv \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \end{pmatrix} \equiv \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix}, \quad \tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1 & \tilde{\lambda}_2 & \tilde{\lambda}_3 & \cdots & \tilde{\lambda}_n \end{pmatrix} \equiv \begin{pmatrix} \tilde{\lambda}^1 \\ \tilde{\lambda}^2 \end{pmatrix}
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The “two–plane” \(\lambda\):

the span of 2 vectors \(\lambda^{\alpha} \in \mathbb{C}^n\)
The Grassmannian Geometry of Kinematical Constraints

Thus, all the kinematical data can be described by a pair of $\lambda \equiv (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$ and $\tilde{\lambda} \equiv (\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \ldots, \tilde{\lambda}_n)$ matrices:

$$\lambda \equiv (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \equiv \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} \quad \tilde{\lambda} \equiv (\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \ldots, \tilde{\lambda}_n) \equiv \begin{pmatrix} \tilde{\lambda}^1 \\ \tilde{\lambda}^2 \end{pmatrix}$$

writing $\lambda_a \in \mathbb{C}^2$ for a column, $\lambda^\alpha \in \mathbb{C}^n$ for a row.

- Because Lorentz transformations mix the rows of each matrix, $\lambda^\alpha, \tilde{\lambda}^\dot{\alpha}$, and the little group allows for rescaling, the invariant content of the data is:

The “two–plane” $\lambda$:

the span of 2 vectors $\lambda^\alpha \in \mathbb{C}^n$
The Grassmannian Geometry of Kinematical Constraints

Thus, all the kinematical data can be described by a pair of \((2 \times n)\) matrices:

\[
\lambda \equiv \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \end{pmatrix} \equiv \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} \quad \tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1 & \tilde{\lambda}_2 & \tilde{\lambda}_3 & \cdots & \tilde{\lambda}_n \end{pmatrix} \equiv \begin{pmatrix} \tilde{\lambda}^1 \\ \tilde{\lambda}^2 \end{pmatrix}
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The Grassmannian Geometry of Kinematical Constraints

Thus, all the kinematical data can be described by a pair of \((2 \times n)\) matrices:

\[
\lambda \equiv \left( \begin{array}{c}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\vdots \\
\lambda_n
\end{array} \right) \equiv \left( \begin{array}{c}
\lambda^1 \\
\lambda^2
\end{array} \right) \quad \tilde{\lambda} \equiv \left( \begin{array}{c}
\tilde{\lambda}_1 \\
\tilde{\lambda}_2 \\
\tilde{\lambda}_3 \\
\vdots \\
\tilde{\lambda}_n
\end{array} \right) \equiv \left( \begin{array}{c}
\tilde{\lambda}^1 \\
\tilde{\lambda}^2
\end{array} \right)
\]

writing \(\lambda_a \in \mathbb{C}^2\) for a column, \(\lambda^\alpha \in \mathbb{C}^n\) for a row.

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The Grassmannian Geometry of Kinematical Constraints

Thus, all the kinematical data can be described by a pair of \((2 \times n)\) matrices:

\[
\lambda \equiv \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \end{pmatrix} \equiv \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} \quad \text{and} \quad \tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1 & \tilde{\lambda}_2 & \tilde{\lambda}_3 & \cdots & \tilde{\lambda}_n \end{pmatrix} \equiv \begin{pmatrix} \tilde{\lambda}^1 \\ \tilde{\lambda}^2 \end{pmatrix}
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- Because Lorentz transformations mix the rows of each matrix, \(\lambda^\alpha\), \(\tilde{\lambda}^{\dot{\alpha}}\), and the little group allows for rescaling, the invariant content of the data is:

The Grassmannian \(G(k, n)\): the span of \(k\) vectors in \(\mathbb{C}^n\)
The Grassmannian Geometry of Kinematical Constraints

Thus, all the kinematical data can be described by a pair of \((2 \times n)\) matrices:

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\lambda \equiv \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \end{pmatrix} \equiv \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} \quad \tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1 & \tilde{\lambda}_2 & \tilde{\lambda}_3 & \cdots & \tilde{\lambda}_n \end{pmatrix} \equiv \begin{pmatrix} \tilde{\lambda}^1 \\ \tilde{\lambda}^2 \end{pmatrix}
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**The Grassmannian** \(G(k, n)\):

the *span* of \(k\) vectors in \(\mathbb{C}^n\)
The Grassmannian Geometry of Kinematical Constraints

Thus, all the kinematical data can be described by a pair of \((2 \times n)\) matrices:

\[
\lambda \equiv \left( \lambda_1 \; \lambda_2 \; \lambda_3 \; \cdots \; \lambda_n \right) \equiv \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix} \quad \tilde{\lambda} \equiv \left( \tilde{\lambda}_1 \; \tilde{\lambda}_2 \; \tilde{\lambda}_3 \; \cdots \; \tilde{\lambda}_n \right) \equiv \begin{pmatrix} \tilde{\lambda}^1 \\ \tilde{\lambda}^2 \end{pmatrix}
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The Grassmannian \(G(k, n)\):

- the span of \(k\) vectors in \(\mathbb{C}^n\)

- Momentum conservation:
The **Grassmannian** Geometry of Kinematical Constraints

Thus, all the kinematical data can be described by a pair of \((2 \times n)\) matrices:

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\]

writing \(\lambda_d \in \mathbb{C}^2\) for a column, \(\lambda^\alpha \in \mathbb{C}^n\) for a row.

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- **Momentum conservation:**
  (taking all the momenta to be incoming)
The **Grassmannian Geometry** of Kinematical Constraints

Thus, all the kinematical data can be described by a pair of $(2 \times n)$ matrices:

$$
\lambda \equiv \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \end{pmatrix} \equiv \begin{pmatrix} \lambda_1^1 \\ \lambda_2 \end{pmatrix} \quad \tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1 & \tilde{\lambda}_2 & \tilde{\lambda}_3 & \cdots & \tilde{\lambda}_n \end{pmatrix} \equiv \begin{pmatrix} \tilde{\lambda}_1^1 \\ \tilde{\lambda}_2 \end{pmatrix}
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- **Momentum conservation:**
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$$
\delta^4 \left( \sum_a p_a^\mu \right)
$$

---

**Amplitudes 2018 Summer School**  QMAP, University of California, Davis

**Part I: The Vernacular of the S-Matrix**
The Grassmannian Geometry of Kinematical Constraints

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\[
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The Grassmannian Geometry of Kinematical Constraints

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The Grassmannian $G(k, n)$:

the span of $k$ vectors in $\mathbb{C}^n$
The Grassmannian Geometry of Kinematical Constraints

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\tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1 & \tilde{\lambda}_2 & \tilde{\lambda}_3 & \cdots & \tilde{\lambda}_n \end{pmatrix} \equiv \begin{pmatrix} \tilde{\lambda}^1 \\ \tilde{\lambda}^2 \end{pmatrix}
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\delta^4 \left( \sum_a p^\mu_a \right) = \delta^{2 \times 2} \left( \sum_a \lambda^\alpha_a \tilde{\lambda}^{\dot{\alpha}}_a \right) \equiv \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda})
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance uniquely fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance uniquely fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[ f(\lambda_1, \lambda_2, \lambda_3) = \langle 23 \rangle^4 \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \delta^2 \times \left( \lambda \cdot \tilde{\lambda} \right) \]

\[ \tilde{\lambda} \equiv \left( \langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \right) \]

\[ \lambda \perp \equiv \left( \langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \right) \supset \lambda \]

\[ h_1 + h_2 + h_3 \leq 0 \]

\[ h_1 + h_2 + h_3 \geq 0 \]

\[ \rightarrow \langle a b \rangle \rightarrow O(\epsilon) \]

\[ O(\epsilon - (h_1 + h_2 + h_3)) \]

\[ \rightarrow \left[ a b \right] \rightarrow O(\epsilon) \]

\[ O(\epsilon - (h_1 + h_2 + h_3)) \]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance \textbf{uniquely} fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[ h_1 = f(\lambda_1 \tilde{\lambda}_1, \lambda_2 \tilde{\lambda}_2, \lambda_3 \tilde{\lambda}_3) \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[ f(\lambda_1 \tilde{\lambda}_1, \lambda_2 \tilde{\lambda}_2, \lambda_3 \tilde{\lambda}_3) \delta^2 \times 2 (\lambda \cdot \tilde{\lambda}) \]

\[ \lambda \equiv \begin{pmatrix} \lambda_1^1 \\ \lambda_1^2 \\ \lambda_1^3 \end{pmatrix} \]

\[ \tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1^1 \\ \tilde{\lambda}_1^2 \\ \tilde{\lambda}_1^3 \end{pmatrix} \]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
f(\lambda_1 \tilde{\lambda}_1, \lambda_2 \tilde{\lambda}_2, \lambda_3 \tilde{\lambda}_3) \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda})
\]

\[
\lambda \equiv \begin{pmatrix} \lambda_1^1 \\ \lambda_2^1 \\ \lambda_3^1 \end{pmatrix}
\]

\[
\tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1^1 \\ \tilde{\lambda}_2^1 \\ \tilde{\lambda}_3^1 \end{pmatrix}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance uniquely fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
h_1 = f(\lambda_1 \tilde{\lambda}_1, \lambda_2 \tilde{\lambda}_2, \lambda_3 \tilde{\lambda}_3) \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \Rightarrow \begin{cases} 
\lambda \equiv \begin{pmatrix} \lambda_1^1 \\ \lambda_2^1 \\ \lambda_3^1 
\end{pmatrix} \\
\tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1^1 \\ \tilde{\lambda}_2^1 \\ \tilde{\lambda}_3^1 
\end{pmatrix}
\end{cases}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance uniquely fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
\langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \\
\lambda^\perp \equiv \begin{pmatrix}
\langle 23 \rangle \\
\langle 31 \rangle \\
\langle 12 \rangle
\end{pmatrix}
\end{align*}
\]

\[
\lambda \equiv \begin{pmatrix}
\lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\
\lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\
\lambda_1^3 & \lambda_2^3 & \lambda_3^3
\end{pmatrix}
\]

\[
\tilde{\lambda} \equiv \begin{pmatrix}
\tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \tilde{\lambda}_3^1 \\
\tilde{\lambda}_1^2 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \\
\tilde{\lambda}_1^3 & \tilde{\lambda}_2^3 & \tilde{\lambda}_3^3
\end{pmatrix}
\]

\[
\tilde{\lambda}^\perp \equiv \begin{pmatrix}
[23] \\
[31] \\
[12]
\end{pmatrix}
\]
Building Blocks: the S-Matrix for Three Massless Particles

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Building Blocks: the S-Matrix for Three Massless Particles

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\[
h_1 = f(\lambda_1 \tilde{\lambda}_1, \lambda_2 \tilde{\lambda}_2, \lambda_3 \tilde{\lambda}_3) \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \Rightarrow
\]

\[
\begin{align*}
\lambda & \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\
\lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\
\lambda_1^1 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \\
\tilde{\lambda} & \equiv \begin{pmatrix} \tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \tilde{\lambda}_3^1 \\
\tilde{\lambda}_1^2 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \\
\tilde{\lambda}_1^1 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \end{pmatrix} \\
\lambda^\perp & \equiv \langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \supset \tilde{\lambda}
\end{align*}
\]

or

\[
\begin{align*}
\lambda & \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\
\lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\
\lambda_1^1 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \\
\tilde{\lambda} & \equiv \begin{pmatrix} \tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \tilde{\lambda}_3^1 \\
\tilde{\lambda}_1^2 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \\
\tilde{\lambda}_1^1 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \end{pmatrix} \\
\lambda^\perp & \equiv \langle [23] [31] [12] \rangle \supset \lambda
\end{align*}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance uniquely fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[ \lambda \equiv \begin{pmatrix} \lambda_1^1 \\ \lambda_2^1 \\ \lambda_3^1 \end{pmatrix} \]

or

\[ \tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1^1 \\ \tilde{\lambda}_2^1 \\ \tilde{\lambda}_3^1 \end{pmatrix} \]

\[ \tilde{\lambda}^\perp \equiv ([23] [31] [12]) \supset \lambda \]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance uniquely fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\lambda^\perp \equiv \left( \langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \right) \equiv \tilde{\lambda}
\]

\[
\lambda \equiv \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{pmatrix}
\]

or

\[
\tilde{\lambda} \equiv \begin{pmatrix}
\tilde{\lambda}_1 \\
\tilde{\lambda}_2 \\
\tilde{\lambda}_3
\end{pmatrix}
\]

\[
\tilde{\lambda}^\perp \equiv \left( [23] [31] [12] \right) \equiv \lambda
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance uniquely fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[ f(\lambda_1, \lambda_2, \lambda_3) \]

\[ f(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3) \]

\[ \lambda^\perp \equiv (\langle 23 \rangle \langle 31 \rangle \langle 12 \rangle) \subset \tilde{\lambda} \]

\[ \lambda \equiv \left( \begin{array}{ccc} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{array} \right) \]

or

\[ \tilde{\lambda} \equiv \left( \begin{array}{ccc} \tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \tilde{\lambda}_3^1 \\ \tilde{\lambda}_1^2 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \\ \tilde{\lambda}_1^3 & \tilde{\lambda}_2^3 & \tilde{\lambda}_3^3 \end{array} \right) \]

\[ \tilde{\lambda}^\perp \equiv (\langle 23 \rangle \langle 31 \rangle \langle 12 \rangle) \subset \lambda \]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1}
\]

or

\[
\tilde{\lambda} \equiv \left( \tilde{\lambda}_1^1 \right) \left( \tilde{\lambda}_2^1 \right) \left( \tilde{\lambda}_3^1 \right)
\]

and

\[
\lambda^\perp \equiv \left( \langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \right) \supset \tilde{\lambda}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance uniquely fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[ f(\lambda_1, \lambda_2, \lambda_3) \propto \left[ \begin{array}{c} \langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \end{array} \right] \delta_{2} \times 2 \left( \lambda \cdot \tilde{\lambda} \right) \]

\[ \lambda^\perp \equiv \left( \langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \right) \supset \tilde{\lambda} \]

\[ \lambda \equiv \left( \begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{array} \right) \]

or

\[ \tilde{\lambda} \equiv \left( \begin{array}{ccc} \tilde{\lambda}_1 & \tilde{\lambda}_2 & \tilde{\lambda}_3 \\ \tilde{\lambda}_1 & \tilde{\lambda}_2 & \tilde{\lambda}_3 \\ \tilde{\lambda}_1 & \tilde{\lambda}_2 & \tilde{\lambda}_3 \end{array} \right) \]

\[ \tilde{\lambda}^\perp \equiv \left( \langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \right) \supset \lambda \]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
\langle 12 \rangle^{h_3-h_1-h_2} & \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1} \\
\langle a b \rangle & \to \mathcal{O}(\epsilon) \\
\end{align*}
\]

\[
\begin{align*}
[12]^{h_1+h_2-h_3} & [23]^{h_2+h_3-h_1} [31]^{h_3+h_1-h_2} \\
[a b] & \to \mathcal{O}(\epsilon) \\
\end{align*}
\]

\[
\lambda \equiv \left( \begin{array}{ccc}
\lambda_1 & \lambda_2 & \lambda_3 \\
\end{array} \right)
\]

\[
\lambda^\perp \equiv \left( \begin{array}{c}
\langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \\
\end{array} \right) \hat{\lambda}
\]

or

\[
\tilde{\lambda} \equiv \left( \begin{array}{ccc}
\tilde{\lambda}_1 & \tilde{\lambda}_2 & \tilde{\lambda}_3 \\
\end{array} \right)
\]

\[
\tilde{\lambda}^\perp \equiv \left( \begin{array}{c}
[23] [31] [12] \\
\end{array} \right) \hat{\lambda}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance \textbf{uniquely} fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[ f(\lambda_1, \lambda_2, \lambda_3) \propto \langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \delta^2 \times 2 \left( \lambda \cdot \tilde{\lambda} \right) \]

\[ \lambda^\perp \equiv \left( \langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \right) \tilde{\lambda} \]

\[ \lambda \equiv \left( \begin{array}{ccc} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{array} \right) \]

or

\[ \tilde{\lambda} \equiv \left( \begin{array}{ccc} \tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \tilde{\lambda}_3^1 \\ \tilde{\lambda}_1^2 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \\ \tilde{\lambda}_1^3 & \tilde{\lambda}_2^3 & \tilde{\lambda}_3^3 \end{array} \right) \]

\[ \tilde{\lambda}^\perp \equiv \left( \langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \right) \lambda \]
Building Blocks: the S-Matrix for Three Massless Particles

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\[
\langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1}
\]
\[
h_1 + h_2 + h_3 \leq 0
\]

\[
[12]^{h_1+h_2-h_3} [23]^{h_2+h_3-h_1} [31]^{h_3+h_1-h_2}
\]
\[
h_1 + h_2 + h_3 \geq 0
\]
Building Blocks: the S-Matrix for Three Massless Particles

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\[ h_1 + h_2 + h_3 \leq 0 \]

\[ h_1 + h_2 + h_3 \geq 0 \]
Building Blocks: the S-Matrix for Three Massless Particles

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\[ \langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1} \]
\[ h_1 + h_2 + h_3 \leq 0 \]

\[ [12]^{h_1+h_2-h_3} [23]^{h_2+h_3-h_1} [31]^{h_3+h_1-h_2} \]
\[ h_1 + h_2 + h_3 \geq 0 \]

\[ \lambda \parallel \equiv \left( \langle 23 \rangle \langle 31 \rangle \langle 12 \rangle \right) \cdot \lambda \]
\[ \lambda \equiv \left( \begin{array}{ccc} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{array} \right) \]

or

\[ \lambda \parallel \equiv \left( \begin{array}{ccc} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{array} \right) \cdot \lambda \]

\[ \tilde{\lambda} \parallel \equiv \left( \begin{array}{ccc} \tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \tilde{\lambda}_3^1 \\ \tilde{\lambda}_1^2 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \\ \tilde{\lambda}_1^3 & \tilde{\lambda}_2^3 & \tilde{\lambda}_3^3 \end{array} \right) \cdot \lambda \]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance \textbf{uniquely} fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
\langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1} \\
& \propto \langle 12 \rangle^{h_1+h_2-h_3} \langle 23 \rangle^{h_2+h_3-h_1} \langle 31 \rangle^{h_3+h_1-h_2} \\
& h_1 + h_2 + h_3 \leq 0 \\
& h_1 + h_2 + h_3 \geq 0
\end{align*}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
1^+ &\quad \propto \langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1} \\
&\quad \quad \quad h_1 + h_2 + h_3 \leq 0 \\
3^- &\quad \propto \langle 12 \rangle^{h_1+h_2-h_3} \langle 23 \rangle^{h_2+h_3-h_1} \langle 31 \rangle^{h_3+h_1-h_2} \\
&\quad \quad \quad h_1 + h_2 + h_3 \geq 0
\end{align*}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
1^+ = \frac{\langle 2 \ 3 \rangle^4}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \langle 3 \ 1 \rangle} \delta^{2 \times 2} (\lambda \cdot \bar{\lambda})
\]

\[
1^- = \frac{[2 \ 3]^4}{[1 \ 2] [2 \ 3] [3 \ 1]} \delta^{2 \times 2} (\lambda \cdot \bar{\lambda})
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance uniquely fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
1^+ & \quad = \quad \frac{\langle 2 \, 3 \rangle^4}{\langle 1 \, 2 \rangle \langle 2 \, 3 \rangle \langle 3 \, 1 \rangle} \quad \delta^{2 \times 2} (\lambda \cdot \bar{\lambda}) \\
1^- & \quad = \quad \frac{[2 \, 3]^4}{[1 \, 2] \, [2 \, 3] \, [3 \, 1]} \quad \delta^{2 \times 2} (\lambda \cdot \bar{\lambda})
\end{align*}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance uniquely fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
1 \rightarrow 2 \rightarrow 3 & = \frac{\langle 2 \, 3 \rangle^4}{\langle 1 \, 2 \rangle \langle 2 \, 3 \rangle \langle 3 \, 1 \rangle} \delta^{2 \times 2} (\lambda \cdot \bar{\lambda}) \\
1 \rightarrow 3 \rightarrow 2 & = \frac{[2 \, 3]^4}{[1 \, 2] [2 \, 3] [3 \, 1]} \delta^{2 \times 2} (\lambda \cdot \bar{\lambda})
\end{align*}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance \textit{uniquely} fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
\begin{align*}
1 & \quad = \quad \frac{\langle 2 \ 3 \rangle^4}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \langle 3 \ 1 \rangle} \quad \delta^{2 \times 2} (\lambda \cdot \bar{\lambda}) \equiv A_3(+, -, -) \\
1 & \quad = \quad \frac{[2 \ 3]^4}{[1 \ 2] \ [2 \ 3] \ [3 \ 1]} \quad \delta^{2 \times 2} (\lambda \cdot \bar{\lambda}) \equiv A_3(-, +, +)
\end{align*}
\end{align*}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
1 & \rightarrow 2 \rightarrow 3 \\
& = \frac{\langle 2 \, 3 \rangle^4}{\langle 1 \, 2 \rangle \langle 2 \, 3 \rangle \langle 3 \, 1 \rangle} \delta^{2 \times 2} (\lambda \cdot \bar{\lambda}) \equiv A_3 (+, -, -) \\
1 & \rightarrow 3 \rightarrow 2 \\
& = \frac{[2 \, 3]^4}{[1 \, 2] [2 \, 3] [3 \, 1]} \delta^{2 \times 2} (\lambda \cdot \bar{\lambda}) \equiv A_3 (-, +, +)
\end{align*}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
1 & \to 2 \to 3 \to 2 \\
= & \frac{\langle 3 1 \rangle \langle 2 3 \rangle^3}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 1 \rangle} \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv A_3 \left( +\frac{1}{2}, -\frac{1}{2}, - \right) \\
\end{align*}
\]

\[
\begin{align*}
1 & \to 3 \\
= & \frac{[3 1][2 3]^3}{[1 2] [2 3] [3 1]} \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv A_3 \left( -\frac{1}{2}, +\frac{1}{2}, + \right)
\end{align*}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
1 & \quad 2 \\
3 & \quad 2
\end{align*}
\]

\[
\frac{\langle 3 \ 1 \rangle \langle 2 \ 3 \rangle^3}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \langle 3 \ 1 \rangle} \quad \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv A_3 \left( +\frac{1}{2}, -\frac{1}{2}, - \right)
\]

\[
\begin{align*}
1 & \quad 3 \\
2 & \quad 3
\end{align*}
\]

\[
\frac{[3 \ 1][2 \ 3]^3}{[1 \ 2][2 \ 3][3 \ 1]} \quad \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv A_3 \left( -\frac{1}{2}, +\frac{1}{2}, + \right)
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
1 & \quad \quad \quad 2 \\
\quad \quad \quad 3 & \quad \quad \quad \quad \quad 2 \\
1 & \quad \quad \quad 3
\end{align*}
\]

\[
\delta^{2\times4} (\lambda \cdot \tilde{\eta}) \frac{\delta^{2\times2} (\lambda \cdot \tilde{\lambda})}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \langle 3 \ 1 \rangle} \equiv A_{3}^{(2)}
\]

\[
\begin{align*}
1 & \quad \quad \quad 2 \\
\quad \quad \quad 3 & \quad \quad \quad \quad \quad 2 \\
1 & \quad \quad \quad 3
\end{align*}
\]

\[
\delta^{1\times4} (\tilde{\lambda} \cdot \tilde{\eta}) \frac{\delta^{2\times2} (\lambda \cdot \tilde{\lambda})}{[1 \ 2] [2 \ 3] [3 \ 1]} \equiv A_{3}^{(1)}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance uniquely fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
1 & \quad 2 \\
\quad & \quad 3 \\
1 & \quad 2 \\
\quad & \quad 3
\end{align*}
\]

\[
\begin{align*}
1 & \quad 2 \\
\quad & \quad 3 \\
1 & \quad 2 \\
\quad & \quad 3
\end{align*}
\]

\[
\begin{align*}
\delta^{2 \times 4} (\lambda \cdot \eta) & \quad \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv A_3^{(2)} \\
\delta^{1 \times 4} (\tilde{\lambda} \cdot \eta) & \quad \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv A_3^{(1)}
\end{align*}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
1 \quad & = \frac{\delta^2 \times 4 (\lambda \cdot \tilde{\eta})}{\langle 1 \quad 2 \rangle \langle 2 \quad 3 \rangle \langle 3 \quad 1 \rangle} \quad \delta^2 \times 2 (\lambda \cdot \tilde{\lambda}) \equiv A_3^{(2)} \\
& = \frac{\delta^1 \times 4 (\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[1 \quad 2] [2 \quad 3] [3 \quad 1]} \quad \delta^2 \times 2 (\lambda \cdot \tilde{\lambda}) \equiv A_3^{(1)}
\end{align*}
\]
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\[
\begin{align*}
1 & \quad 2 \\
3 & \quad 2 & = & \frac{\delta^{2 \times 4} (\lambda \cdot \tilde{\eta})}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \langle 3 \ 1 \rangle} \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv A_3^{(2)} \\
1 & \quad 2 \\
3 & \quad 2 & = & \frac{\delta^{1 \times 4} (\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[1 \ 2 \ 2 \ 3 \ 3 \ 1]} \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \equiv A_3^{(1)}
\end{align*}
\]
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

\[
\begin{align*}
1 & \rightarrow 2 \\
3 & \rightarrow 3 \\
2 & \rightarrow 2 \\
\end{align*}
\]

\[
\begin{align*}
\delta^2 \times 4 \left( \lambda \cdot \tilde{\eta} \right) &= \frac{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 1 \rangle}{\delta^2 \times 2 \left( \lambda \cdot \tilde{\lambda} \right)} \equiv A_3^{(2)} \\
\delta^1 \times 4 \left( \tilde{\lambda} \cdot \tilde{\eta} \right) &= \frac{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 1 \rangle}{\delta^2 \times 2 \left( \lambda \cdot \tilde{\lambda} \right)} \equiv A_3^{(1)} \\
\end{align*}
\]
Amalgamating Diagrams from Three-Particle Amplitudes

On-shell diagrams built out of only three-particle amplitudes are well-defined to all orders of perturbation theory.
Amalgamating Diagrams from Three-Particle Amplitudes

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On-shell diagrams built out of only three-particle amplitudes are well-defined to all orders of perturbation theory, generating a large class of functions:
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Amalgamating Diagrams from Three-Particle Amplitudes

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![Diagram of a three-particle amplitude](image-url)
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On-shell diagrams built out of only three-particle amplitudes are well-defined to all orders of perturbation theory, generating a large class of functions:

\[
\begin{align*}
\left(\epsilon^{abc} \epsilon^{d12} \epsilon^{345} \epsilon^{678} \epsilon^{910} \epsilon^{123} \epsilon^{456} \epsilon^{293} \epsilon^{412} \epsilon^{639} \epsilon^{916} \right)
\end{align*}
\]
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On-shell diagrams built out of only three-particle amplitudes are well-defined to all orders of perturbation theory, generating a large class of functions:

\[
\begin{align*}
&= \left( \langle 91 \rangle \langle 23 \rangle \langle 46 \rangle - \langle 16 \rangle \langle 34 \rangle \langle 29 \rangle \right) \delta^2 \times 4 \left( \lambda \cdot \tilde{\eta} \right) \\
&\quad \times \left( \lambda \cdot \tilde{\lambda} \right) \\
&\quad \times \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle 
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\[
\left(\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 45\rangle \langle 56\rangle \langle 67\rangle \langle 78\rangle \langle 81\rangle \langle 14\rangle \langle 42\rangle \langle 29\rangle \langle 96\rangle \langle 63\rangle \langle 39\rangle \langle 91\rangle \right)_{2} \delta_{2}^{4} \left(\lambda \cdot \tilde{\eta} \right)_{2} \delta_{2}^{4} \left(\lambda \cdot \tilde{\lambda} \right)
\]
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\[
\left( \langle 91 \rangle \langle 23 \rangle \langle 46 \rangle - \langle 16 \rangle \langle 34 \rangle \langle 29 \rangle \right)^2 \delta^2 \times 4 \left( \lambda \cdot \tilde{\eta} \right) \delta^2 \times 2 \left( \lambda \cdot \tilde{\lambda} \right)
\]

\[
\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle
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On-shell diagrams built out of only \textbf{three-particle amplitudes} are well-defined to all orders of perturbation theory, generating a large class of functions:

\[
\begin{align*}
\langle 2 \rangle \langle 3 \rangle \\
\end{align*}
\]
Amalgamating Diagrams from Three-Particle Amplitudes

On-shell diagrams built out of only three-particle amplitudes are well-defined to all orders of perturbation theory, generating a large class of functions:

\[
\frac{1}{2} \delta^2 \times 4 \left( \lambda \cdot \tilde{\eta} \right) \delta^2 \times 2 \left( \lambda \cdot \tilde{\lambda} \right)
\]

\[
\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle
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\[
\begin{align*}
\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle
\end{align*}
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\[
\left(\begin{array}{c}
\langle 1 2 \rangle \\
\langle 2 3 \rangle \\
\langle 3 4 \rangle \\
\langle 4 5 \rangle \\
\langle 5 6 \rangle \\
\langle 6 7 \rangle \\
\langle 7 8 \rangle \\
\langle 8 9 \rangle \\
\langle 9 1 \rangle \\
\langle 1 4 \rangle \\
\langle 2 9 \rangle \\
\langle 3 6 \rangle \\
\langle 4 2 \rangle \\
\langle 5 3 \rangle \\
\langle 6 1 \rangle \\
\langle 7 4 \rangle \\
\langle 8 5 \rangle \\
\langle 9 7 \rangle \\
\end{array}\right) \\
\delta^2 \\
\times \\
4 \\
\left(\begin{array}{c}
\lambda \cdot \tilde{\eta} \\
\delta^2 \\
\times \\
2 \\
\lambda \cdot \tilde{\lambda} \\
\end{array}\right) \\
\left(\begin{array}{c}
\langle 1 2 \rangle \\
\langle 2 3 \rangle \\
\langle 3 4 \rangle \\
\langle 4 5 \rangle \\
\langle 5 6 \rangle \\
\langle 6 7 \rangle \\
\langle 7 8 \rangle \\
\langle 8 9 \rangle \\
\langle 9 1 \rangle \\
\langle 1 4 \rangle \\
\langle 2 9 \rangle \\
\langle 3 6 \rangle \\
\langle 4 2 \rangle \\
\langle 5 3 \rangle \\
\langle 6 1 \rangle \\
\langle 7 4 \rangle \\
\langle 8 5 \rangle \\
\langle 9 7 \rangle \\
\end{array}\right)
\]
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\[
\begin{align*}
\langle 1 & 2 \rangle \langle 2 & 3 \rangle \langle 3 & 4 \rangle \langle 4 & 5 \rangle \langle 5 & 6 \rangle \langle 6 & 7 \rangle \langle 7 & 8 \rangle \langle 8 & 1 \rangle \langle 1 & 4 \rangle \langle 4 & 2 \rangle \langle 2 & 9 \rangle \langle 9 & 6 \rangle \langle 6 & 3 \rangle \langle 3 & 9 \rangle \langle 9 & 1 \rangle \\
\times & \left( \lambda \cdot \tilde{\eta} \right) \delta^2 \times 4 \left( \lambda \cdot \lambda \right)
\end{align*}
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\[
\left( \begin{array}{l}
\langle 1 \rangle \\
\langle 2 \rangle \\
\langle 3 \rangle \\
\langle 4 \rangle \\
\langle 5 \rangle \\
\langle 6 \rangle \\
\langle 7 \rangle \\
\langle 8 \rangle \\
\langle 9 \rangle \\
\langle 10 \rangle \\
\langle 11 \rangle \\
\langle 12 \rangle \\
\end{array} \right)
\]
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\[
\begin{align*}
&\langle 2 \rangle \langle 3 \rangle \\
&\langle 4 \rangle
\end{align*}
\]

\[
\begin{align*}
&\langle 1 \rangle \langle 2 \rangle \\
&\langle 4 \rangle
\end{align*}
\]

\[
\begin{align*}
&\langle 1 \rangle \langle 2 \rangle \\
&\langle 3 \rangle
\end{align*}
\]

\[
\begin{align*}
&\langle 1 \rangle \langle 4 \rangle \\
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\[
\begin{align*}
\langle 91 \rangle \langle 23 \rangle \langle 46 \rangle &- \langle 16 \rangle \langle 34 \rangle \langle 29 \rangle \\
\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle
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\[
\left(\langle 91 \rangle \langle 23 \rangle \langle 46 \rangle - \langle 16 \rangle \langle 34 \rangle \langle 29 \rangle \right)^2 \delta^2_2 
	\times 4 \left(\lambda \cdot \tilde{\eta} \right) 
	\times 2 \left(\lambda \cdot \tilde{\lambda} \right)
\]

\[
\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle
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\[\left(\langle 91 \rangle \langle 23 \rangle \langle 46 \rangle - \langle 16 \rangle \langle 34 \rangle \langle 29 \rangle\right)^2 \delta^2 \times 4 \left(\lambda \cdot \tilde{\eta}\right) \delta^2 \times 2 \left(\lambda \cdot \tilde{\lambda}\right) \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle\]
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On-shell diagrams built out of only three-particle amplitudes are well-defined to all orders of perturbation theory, generating a large class of functions:

\[
\left(\begin{array}{c}
\langle 12 \rangle \\
\langle 23 \rangle \\
\langle 34 \rangle \\
\langle 45 \rangle \\
\langle 56 \rangle \\
\langle 67 \rangle \\
\langle 78 \rangle \\
\langle 81 \rangle \\
\langle 14 \rangle \\
\langle 42 \rangle \\
\langle 29 \rangle \\
\langle 96 \rangle \\
\langle 63 \rangle \\
\langle 39 \rangle \\
\langle 91 \rangle \\
\end{array}\right)
\]
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On-shell diagrams built out of only \textbf{three-particle amplitudes} are well-defined to all orders of perturbation theory, generating a large class of functions:

\[
= \frac{(\langle 91 \rangle \langle 23 \rangle \langle 46 \rangle - \langle 16 \rangle \langle 34 \rangle \langle 29 \rangle)^2 \delta^{2\times4}(\lambda \cdot \tilde{\eta}) \delta^{2\times2}(\lambda \cdot \tilde{\lambda})}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle}
\]
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On-shell diagrams built out of only three-particle amplitudes are well-defined to all orders of perturbation theory, generating a large class of functions:

\[
\left( \langle 91 \rangle \langle 23 \rangle \langle 46 \rangle - \langle 16 \rangle \langle 34 \rangle \langle 29 \rangle \right)^2 \delta^{2 \times 4} (\lambda \cdot \tilde{\eta}) \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda})
\]

\[
\frac{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle}{\langle 91 \rangle \langle 23 \rangle \langle 46 \rangle - \langle 16 \rangle \langle 34 \rangle \langle 29 \rangle}
\]

Part I: The Vernacular of the S-Matrix
Amalgamating Diagrams from Three-Particle Amplitudes

On-shell diagrams built out of only **three-particle amplitudes** are well-defined to all orders of perturbation theory, generating a large class of functions:

\[
\frac{(\langle 91\rangle \langle 23\rangle \langle 46\rangle - \langle 16\rangle \langle 34\rangle \langle 29\rangle)^2 \delta^2 \times 4 (\lambda \cdot \tilde{\eta}) \delta^2 \times 2 (\lambda \cdot \tilde{\lambda})}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 45\rangle \langle 56\rangle \langle 67\rangle \langle 78\rangle \langle 81\rangle \langle 14\rangle \langle 42\rangle \langle 29\rangle \langle 96\rangle \langle 63\rangle \langle 39\rangle \langle 91\rangle}
\]
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$$
= \frac{\langle 91 \rangle \langle 23 \rangle \langle 46 \rangle - \langle 16 \rangle \langle 34 \rangle \langle 29 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle} \delta^2 \times 4 (\lambda \cdot \eta) \delta^2 \times 2 (\lambda \cdot \tilde{\lambda})
$$
Amalgamating Diagrams from Three-Particle Amplitudes

On-shell diagrams built out of only three-particle amplitudes are well-defined to all orders of perturbation theory, generating a large class of functions:

\[
\frac{\langle 91 \rangle \langle 23 \rangle \langle 46 \rangle - \langle 16 \rangle \langle 34 \rangle \langle 29 \rangle )^2 \left( \delta^{2 \times 4} (\lambda \cdot \tilde{\eta}) \delta^{2 \times 2} (\lambda \cdot \tilde{\lambda}) \right)}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \langle 78 \rangle \langle 81 \rangle \langle 14 \rangle \langle 42 \rangle \langle 29 \rangle \langle 96 \rangle \langle 63 \rangle \langle 39 \rangle \langle 91 \rangle}
\]
Building-Up On-Shell Diagrams with “BCFW” Bridges

Very complex on-shell diagrams can be constructed by successively adding “BCFW” bridges to diagrams.
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\[ a \quad b \]

\[ f_0 \]

\[ \ldots \]
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Very complex on-shell diagrams can be constructed by successively adding “BCFW” bridges to diagrams (an extremely useful tool!):

\[ a \quad f_0 \quad b \quad \rightarrow \quad a \quad \text{bridge} \quad b \quad f_0 \quad \ldots \]
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\[
\begin{align*}
\begin{array}{c}
\text{\(a\)} \\
\text{\(\cdots\)} \\
\text{\(f_0\)} \\
\end{array}
\end{align*}
\Rightarrow
\begin{align*}
\begin{array}{c}
\text{\(a\)} \\
\text{\(\text{\(\bullet\)}\)} \\
\text{\(\text{\(\bullet\)}\)} \\
\text{\(f_0\)} \\
\text{\(\cdots\)} \\
\end{array}
\end{align*}
\equiv
\begin{align*}
\begin{array}{c}
\text{\(a\)} \\
\text{\(\cdots\)} \\
\text{\(f\)} \\
\end{array}
\end{align*}
\end{align*}
\]
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Very complex on-shell diagrams can be constructed by successively adding “BCFW” bridges to diagrams (an extremely useful tool!):

Adding the bridge has the effect of shifting the momenta $p_a$ and $p_b$ flowing into the diagram $f_0$ according to:

$$
\lambda_a \mapsto \hat{\lambda}_a = \lambda_a - \alpha \lambda_I \\
\lambda_b \mapsto \hat{\lambda}_b = \lambda_b + \alpha \lambda_I,
$$

introducing a new parameter $\alpha$, in terms of which we may write:

$$
f(..., a, b, ...) = d_{\alpha} f_0(..., \hat{a}, \hat{b}, ...).
$$
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\end{align*}
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Introducing a new parameter $\alpha$, in terms of which we may write:

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$$
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and

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$$
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and

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\[
\begin{align*}
\lambda_a \tilde{\lambda}_a &\mapsto \lambda_{\hat{a}} \tilde{\lambda}_{\hat{a}} = \lambda_a \left( \tilde{\lambda}_a - \alpha \tilde{\lambda}_b \right) \quad \text{and} \quad \lambda_b \tilde{\lambda}_b &\mapsto \lambda_{\hat{b}} \tilde{\lambda}_{\hat{b}} = \lambda_b \tilde{\lambda}_b + \alpha \lambda_a \tilde{\lambda}_b ,
\end{align*}
\]
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where $\alpha$ is a new parameter.
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$$f(\ldots, a, b, \ldots) = \frac{d\alpha}{\alpha} f_0(\ldots, \hat{a}, \hat{b}, \ldots)$$
The Analytic Bootstrap: All-Loop Recursion Relations

Consider adding a BCFW bridge to the full $n$-particle scattering amplitude
The Analytic Bootstrap: All-Loop Recursion Relations

Consider adding a BCFW bridge to the full $n$-particle scattering amplitude:

$A_n$
The Analytic Bootstrap: All-Loop Recursion Relations

Consider adding a BCFW bridge to the full $n$-particle scattering amplitude:

\[ A_n \Rightarrow \hat{A}_n(\alpha) \]
The Analytic Bootstrap: All-Loop Recursion Relations

Consider adding a BCFW bridge to the full $n$-particle scattering amplitude, the undeformed amplitude $A_n$ is recovered as the residue about $\alpha = 0$:

\[ A_n = \hat{A}_n(\alpha) \rho_{\alpha = 0} \]

We can use Cauchy's theorem to trade the residue about $\alpha = 0$ for (minus) the sum of residues away from the origin—these come in two types: factorization-channels and forward-limits.
The Analytic Bootstrap: All-Loop Recursion Relations

Consider adding a BCFW bridge to the full $n$-particle scattering amplitude. The undeformed amplitude $A_n$ is recovered as the residue about $\alpha = 0$:

$$A_n = \hat{A}_n(\alpha \to 0) \propto \oint_{\alpha=0} d\alpha \hat{A}_n(\alpha)$$

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The Analytic Bootstrap: All-Loop Recursion Relations

Consider adding a BCFW bridge to the full $n$-particle scattering amplitude; the \textit{undeformed} amplitude $A_n$ is recovered as the \textbf{residue} about $\alpha = 0$:

$$A_n = \hat{A}_n(\alpha \to 0) \propto \oint_{\alpha=0} \frac{d\alpha}{\alpha} \hat{A}_n(\alpha)$$

We can use \textbf{Cauchy’s theorem} to trade the residue about $\alpha = 0$ for (minus) the sum of residues away from the origin:

\[\begin{array}{c}
\begin{array}{c}
A_n \\
1 \quad n
\end{array}
\Rightarrow
\begin{array}{c}
\hat{A}_n(\alpha) \\
1 \quad \alpha \quad n
\end{array}
\end{array}\]
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$$A_n = \hat{A}_n(\alpha \to 0) \propto \int \frac{d\alpha}{\alpha} \hat{A}_n(\alpha)$$

We can use Cauchy’s theorem to trade the residue about $\alpha = 0$ for (minus) the sum of residues away from the origin:
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\[
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\]

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Consider adding a BCFW bridge to the full $n$-particle scattering amplitude
the undeformed amplitude $A_n$ is recovered as the residue about $\alpha = 0$:

$$A_n = \hat{A}_n(\alpha \to 0) \propto \oint_{\alpha=0} d\alpha \frac{\hat{A}_n(\alpha)}{\alpha}$$

We can use Cauchy’s theorem to trade the residue about $\alpha = 0$ for (minus) the sum of residues away from the origin:
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Consider adding a BCFW bridge to the full \( n \)-particle scattering amplitude. The undeformed amplitude \( A_n \) is recovered as the residue about \( \alpha = 0 \):

\[
A_n = \hat{A}_n(\alpha \to 0) \propto \oint_{\alpha = 0} d\alpha \, \hat{A}_n(\alpha)
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The Analytic Bootstrap: All-Loop Recursion Relations

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![Diagram](image)

$\hat{1}$ $\hat{n}$
The Analytic Bootstrap: All-Loop Recursion Relations

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The Analytic Bootstrap: All-Loop Recursion Relations

\[ A_1^n = \sum_{L,R} A_{1}^{L,R} + A_{1}^{L-1} \]
The Analytic Bootstrap: All-Loop Recursion Relations

Diagrams are characterized by ‘m’—the number of “minus-helicity” gluons:

\[
A^\ell_n = \sum_{L,R} L \rightarrow R + A^{\ell-1}_{n+2}
\]
The Analytic Bootstrap: All-Loop Recursion Relations

Diagrams are characterized by ‘$m$’—the number of “minus-helicity” gluons:

$$m \equiv 2n_B + n_W - n_I.$$
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m_L + m_R = m + 1.
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\[ A_n^\ell = \sum_{L,R} L + R + A_{n+2}^{\ell-1} \]
The BCFW recursion relations realize an incredible fantasy: they \textit{directly} produces the \textbf{Parke-Taylor} formula for all amplitudes with $m = 2$, $A_n^{(2)}$!
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The only (non-vanishing) contribution to \( A_n^{(2)} \) is \( A_{n-1}^{(2)} \otimes A_3^{(1)} \):
Exempli Gratia: On-Shell Representations of Amplitudes

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\[
A_4^{(2)} = \]

\[\text{Amplitudes 2018 Summer School  QMAP, University of California, Davis} \]

Part I: The Vernacular of the S-Matrix
Exempli Gratia: On-Shell Representations of Amplitudes

The BCFW recursion relations realize an incredible fantasy: they directly produces the Parke-Taylor formula for all amplitudes with $m=2, \mathcal{A}_n^{(2)}$! The only (non-vanishing) contribution to $\mathcal{A}_n^{(2)}$ is $\mathcal{A}_{n-1}^{(2)} \otimes \mathcal{A}_3^{(1)}$:

$$\mathcal{A}_4^{(2)} = \begin{array}{c} \overset{2}{\circ} & \overset{3}{\circ} \\ \overset{1}{\circ} & \overset{4}{\circ} \end{array} + \begin{array}{c} \overset{2}{\circ} & \overset{3}{\circ} \\ \overset{1}{\circ} & \overset{4}{\circ} \end{array}$$
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$$A_4^{(2)} = \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array}
\end{array}$$
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\[ A_4^{(2)} = \begin{array}{c}
\text{Diagram 1} \\
1 \quad 2 \quad 3 \quad 4
\end{array} \quad = \quad \begin{array}{c}
\text{Diagram 2} \\
1 \quad 2 \quad 3 \quad 4
\end{array} \]
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\[
A_4^{(2)} = A_3^{(1)} \otimes A_3^{(1)} = A_5^{(2)} =
\]

Exempli Gratia: On-Shell Representations of Loop-Amplitude Integrands
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Exempli Gratia: On-Shell Representations of Loop-Amplitude Integrands
**Exempli Gratia: On-Shell Representations of Amplitudes**

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\begin{array}{c}
\text{2} \\
\text{3}
\end{array} \\
\begin{array}{c}
\text{1} \\
\text{4}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{2} \\
\text{3}
\end{array} \\
\begin{array}{c}
\text{1} \\
\text{4}
\end{array}
\end{array}$$

$$\mathcal{A}_5^{(2)} = \begin{array}{c}
\begin{array}{c}
\text{2} \\
\text{4}
\end{array} \\
\begin{array}{c}
\text{3} \\
\text{1}
\end{array} \\
\begin{array}{c}
\text{5}
\end{array}
\end{array}$$

$$\mathcal{A}_6^{(2)} = \begin{array}{c}
\begin{array}{c}
\text{2} \\
\text{4}
\end{array} \\
\begin{array}{c}
\text{3} \\
\text{1}
\end{array} \\
\begin{array}{c}
\text{5}
\end{array}
\end{array}$$
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\[
\mathcal{A}_4^{(2)} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
\begin{array}{c}
4
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
\begin{array}{c}
4
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
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The BCFW recursion relations realize an incredible fantasy: they directly produces the Parke-Taylor formula for all amplitudes with \( m = 2 \), \( A_n^{(2)} \)!

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\[
A_4^{(2)} = A_3^{(1)} = A_2^{(2)}
\]

\[
A_5^{(2)} = A_4^{(2)} = A_3^{(1)}
\]

\[
A_6^{(2)} = A_5^{(2)} = A_4^{(2)}
\]
Exempli Gratia: On-Shell Representations of Amplitudes

The BCFW recursion relations realize an incredible fantasy: they directly produces the Parke-Taylor formula for all amplitudes with $m=2$, $A_n^{(2)}$! The only (non-vanishing) contribution to $A_n^{(2)}$ is $A_{n-1}^{(2)} \otimes A_3^{(1)}$:

\[
\begin{align*}
A_4^{(2)} &= \begin{array}{c} 2 \\ 3 \end{array} \begin{array}{c} 1 \\ 4 \end{array} = \begin{array}{c} 2 \\ 3 \end{array} \begin{array}{c} 1 \\ 4 \end{array} \\
A_5^{(2)} &= \begin{array}{c} 3 \\ 4 \end{array} \begin{array}{c} 1 \\ 5 \end{array} = \begin{array}{c} 3 \\ 4 \end{array} \begin{array}{c} 1 \\ 5 \end{array} \\
A_6^{(2)} &= \begin{array}{c} 4 \\ 5 \end{array} \begin{array}{c} 1 \\ 6 \end{array} = \begin{array}{c} 4 \\ 5 \end{array} \begin{array}{c} 1 \\ 6 \end{array}
\end{align*}
\]
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\[
A_4^{(2)} = A_5^{(2)} = A_6^{(2)} = A_{n-1}^{(2)} \otimes A_3^{(1)}
\]
Exempi Gratia: On-Shell Representations of Amplitudes

The BCFW recursion relations realize an incredible fantasy: they directly produces the Parke-Taylor formula for all amplitudes with $m = 2, A_n^{(2)}$! And it generates very concise formulae for all other amplitudes.
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\[
A_6^{(3)} = A_5^{(3)} + A_4^{(3)} + A_3^{(3)}
\]
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The BCFW recursion relations realize an incredible fantasy: they directly produces the **Parke-Taylor** formula for all amplitudes with \( m = 2, A_n^{(2)} \)!

And it generates **very concise** formulae for all other amplitudes—e.g. \( A_6^{(3)} \):

\[
A_6^{(3)} = 2 \cdot A_5^{(3)} + 2 \cdot A_4^{(2)} + A_4^{(2)}
\]

Observations regarding recursed representations of scattering amplitudes:
- varying recursion 'schema' can generate many 'BCFW formulae'
- on-shell diagrams can often be related in surprising ways
- Is there any way to invariantly characterize the on-shell functions associated with on-shell diagrams?
The BCFW recursion relations realize an incredible fantasy: they directly produces the Parke-Taylor formula for all amplitudes with $m = 2$, $A_n^{(2)}$!

And it generates very concise formulae for all other amplitudes—e.g. $A_6^{(3)}$:

$$A_6^{(3)} = A_4^{(2)} + A_4^{(2)}$$
The BCFW recursion relations realize an incredible fantasy: they **directly** produces the **Parke-Taylor** formula for all amplitudes with \( m = 2, A_n^{(2)} \)!

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\[
A_6^{(3)} = A_4^{(2)} + A_4^{(2)} + A_5^{(2)}
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The BCFW recursion relations realize an incredible fantasy: they **directly** produces the **Parke-Taylor** formula for all amplitudes with $m=2, A_n^{(2)}$!

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$$A_6^{(3)} = 3 + 4 + 5$$

Observations regarding recursed representations of scattering amplitudes:

- varying recursion 'schema' can generate many 'BCFW formulae'
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Is there any way to invariantly characterize the on-shell functions associated with on-shell diagrams?
Exempi Gratia: On-Shell Representations of Amplitudes

The BCFW recursion relations realize an incredible fantasy: they **directly** produces the Parke-Taylor formula for all amplitudes with $m = 2$, $A_n^{(2)}$!

And it generates **very concise** formulae for all other amplitudes—e.g. $A_6^{(3)}$:

\[
A_6^{(3)} = A_1^{(3)} + A_2^{(3)} + A_3^{(3)}
\]
The BCFW recursion relations realize an incredible fantasy: they **directly** produces the **Parke-Taylor** formula for all amplitudes with $m = 2$, $A_n^{(2)}$!

And it generates **very concise** formulae for all other amplitudes—e.g. $A_6^{(3)}$:

\[
A_6^{(3)} = A_4^{(2)} + \text{(other diagrams)}
\]
The BCFW recursion relations realize an incredible fantasy: they directly produces the Parke-Taylor formula for all amplitudes with $m = 2$, $A^{(2)}_n$!

And it generates very concise formulae for all other amplitudes—e.g. $A^{(3)}_6$:

$$A^{(3)}_6 =$$

\[
\begin{align*}
3 & \quad 4 \\
2 & \quad 5 \\
1 & \quad 6 
\end{align*}
\]

$$ + $$

\[
\begin{align*}
3 & \quad 4 \\
2 & \quad 5 \\
1 & \quad 6 
\end{align*}
\]

$$ + $$

\[
\begin{align*}
3 & \quad 4 \\
2 & \quad 5 \\
1 & \quad 6 
\end{align*}
\]
The BCFW recursion relations realize an incredible fantasy: they directly produces the Parke-Taylor formula for all amplitudes with $m=2$, $A_n^{(2)}$! And it generates very concise formulae for all other amplitudes—e.g. $A_6^{(3)}$:

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\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\begin{array}{c}
4
\end{array}
\begin{array}{c}
5
\end{array}
\begin{array}{c}
2
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
6
\end{array}
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\begin{array}{c}
4
\end{array}
\begin{array}{c}
5
\end{array}
\begin{array}{c}
2
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
6
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\begin{array}{c}
4
\end{array}
\begin{array}{c}
5
\end{array}
\begin{array}{c}
2
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
6
\end{array}
\end{array}
\end{array} \end{array}$$

Observations regarding recursed representations of scattering amplitudes:
The BCFW recursion relations realize an incredible fantasy: they \textbf{directly} produces the \textbf{Parke-Taylor} formula for all amplitudes with \( m = 2, A_n^{(2)} \)!

And it generates \textbf{very concise} formulae for all other amplitudes—e.g. \( A_6^{(3)} \):

\[
A_6^{(3)} = \sum \text{Amplitudes of Diagrams}
\]

Observations regarding recursed representations of scattering amplitudes:
- varying recursion ‘schema’ can generate \textit{many} ‘BCFW formulae’
The BCFW recursion relations realize an incredible fantasy: they **directly** produces the **Parke-Taylor** formula for all amplitudes with $m=2$, $A_n^{(2)}$!

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\mathcal{A}_6^{(3)} = \mathcal{A}_{126}^{(3)} \quad + \quad \mathcal{A}_{125}^{(3)} \quad + \quad \mathcal{A}_{16}^{(3)}
\]

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**Exempli Gratia: On-Shell Representations of Amplitudes**

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\[
A_6^{(3)} = \begin{align*}
&\begin{array}{c}
\text{3} \\
\text{4} \\
\text{5}
\end{array} + \begin{array}{c}
\text{2} \\
\text{1} \\
\text{6}
\end{array} + \begin{array}{c}
\text{2} \\
\text{3} \\
\text{4}
\end{array} + \begin{array}{c}
\text{1} \\
\text{6} \\
\text{5}
\end{array}
\end{align*}
\]

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Is there any way to invariantly characterize the on-shell functions associated with on-shell diagrams?
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On-Shell Recursion of Loop-Amplitude Integrands

Let’s look at an example of how loop amplitudes are represented by recursion.

\[ A(2), 0 \times \int_{\ell \in \mathbb{R}^3, 1} \left( \frac{2}{\ell^2 (\ell^* \ell)^2} \right)^4 \int_{\log(\ell^2 (\ell^* \ell)^2)} \frac{4 \ell (p_1 + p_2)}{(p_3 + p_4)^2} \left( \frac{\ell^2 (\ell^* \ell)^2}{\ell^2 (\ell^* \ell)^2} \right)^4 \int_{\log(\ell^2 (\ell^* \ell)^2)} \frac{4 \ell (p_1 + p_2)}{(p_3 + p_4)^2} \left( \frac{\ell^2 (\ell^* \ell)^2}{\ell^2 (\ell^* \ell)^2} \right)^4 \]
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\[ A^n_1 \ldots n = \sum_{L,R} L + R + A^{n+2}_{n+2} \]
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Let’s look at an example of how loop amplitudes are represented by recursion. For $A_4^{(2),1}$, the only terms come from the ‘forward limit’ of the tree $A_6^{(3),0}$:

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For $A^{(2),1}_4$, the only terms come from the ‘forward limit’ of the tree $A^{(3),0}_6$:

$$A^{(2),1}_4 = A^{(3),0}_6 \times \int_{\ell \in \mathbb{R}^3, 1} d \log \left( \frac{\ell^2}{(\ell + p_1)^2} \right) d \log \left( \frac{\ell^2}{(\ell + p_1 + p_2)^2} \right) d \log \left( \frac{\ell^2}{(\ell + p_4)^2} \right)$$
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$$
\int d^4 \ell \quad \leftrightarrow \quad \int \frac{d^2 \lambda_I d^2 \tilde{\lambda}_I}{\text{vol}(GL_1)} d\alpha \langle I1 \rangle [nI] \\
\ell \equiv (\lambda_I \tilde{\lambda}_I + \alpha \lambda_1 \tilde{\lambda}_4) \in \mathbb{R}^{3,1}
$$

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\]

\[
\ell \equiv (\lambda_I \tilde{\lambda}_I + \alpha \lambda_I \lambda_4) \in \mathbb{R}^{3,1}
\]

\[
A_{4}^{(2),0} \times \int d\log \left( \frac{\ell^2}{(\ell - \ell^*)^2} \right) d\log \left( \frac{(\ell + p_1)^2}{(\ell - \ell^*)^2} \right) d\log \left( \frac{(\ell + p_1 + p_2)^2}{(\ell - \ell^*)^2} \right) d\log \left( \frac{(\ell - p_4)^2}{(\ell - \ell^*)^2} \right)
\]

\[
= A_{4}^{(2),0} \times \int d^4 \ell \frac{(p_1 + p_2)^2 (p_3 + p_4)^2}{\ell^2 (\ell + p_1)^2 (\ell + p_1 + p_2)^2 (\ell - p_4)^2}
\]