Lectures on Superstring Amplitudes

Part 3: Low energy effective interactions, Modular graph functions

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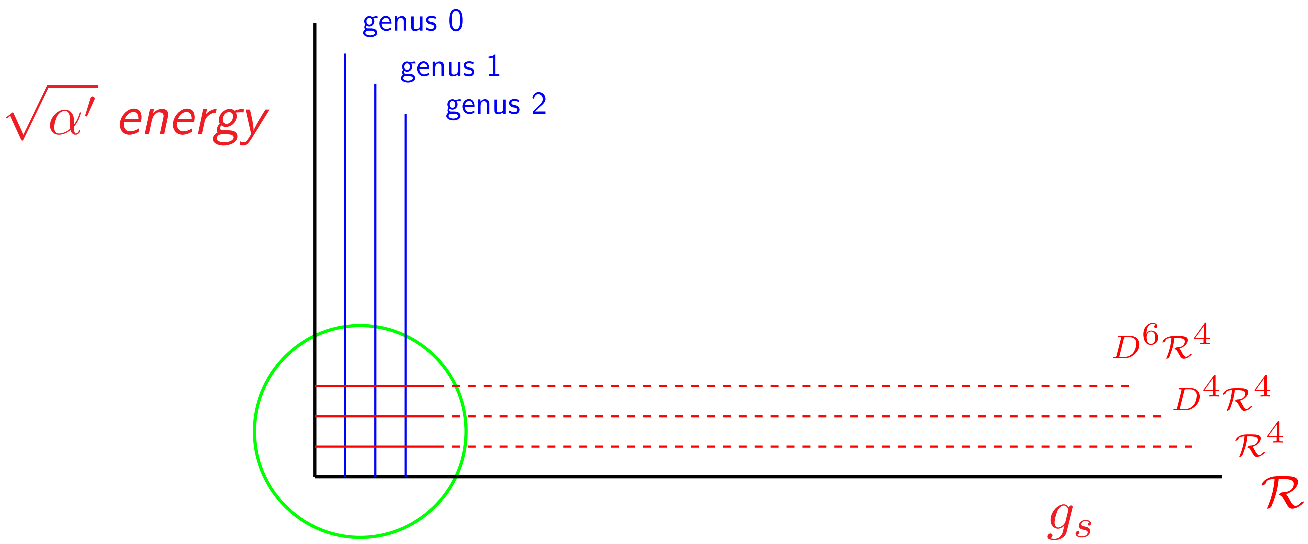
Superstring Perturbation Theory and Supergravity



- **Superstring perturbation theory in powers of the string coupling** g_s
  - holds for weak coupling g_s
  - and for all energies
- **Classical supergravity** "$\mathcal{R}$"
  - leading low energy expansion of string theory
  - holds for all couplings g_s
- **String induced effective interactions** $\mathcal{R}^4, D^4\mathcal{R}^4, D^6\mathcal{R}^4$
  - Evaluated in perturbation theory for $g_s \ll 1$

Low energy expansion of tree-level amplitudes

- **Closed superstring tree-level four-graviton amplitude**

$$\mathcal{A}^{(0)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = \frac{1}{g_s^2} \frac{\mathcal{R}^4}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} \qquad s_{ij} = -\frac{\alpha'}{4}(k_i+k_j)^2$$

  – $\mathcal{R}$ symbolically stands for the Weyl tensor
  – $\mathcal{R}^4$ symbolically stands for a scalar contraction dictated by supersymmetry

- **At low energy** $|s_{ij}| \ll 1$
  – massless string exchanges produce non-local contributions;
  – massive string exchanges produce local effective interactions
  – string-induced corrections to supergravity; eg. in Type II

$$\frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2+t^2+u^2) + 2\zeta(3)^2 stu + \frac{1}{2}\zeta(7)(s^2+t^2+u^2)^2 + \cdots$$

massless $\quad \mathcal{R}^4 \quad D^4\mathcal{R}^4 \quad D^6\mathcal{R}^4 \quad D^8\mathcal{R}^4$

  – $D^{2k}\mathcal{R}^4$ contraction of covariant derivatives D and $\mathcal{R}^4$

Effective interactions from Type IIB superstrings

- $SL(2,\mathbb{Z})$**-duality symmetry of Type IIB superstrings**
  - requires effective interactions to be $SL(2,\mathbb{Z})$-invariant;
  - Einstein frame metric G_E and $\mathcal{R}_E^4$ invariant under $SL(2,\mathbb{Z})$
  - combine axion χ dilaton Φ in $\rho = \chi + ie^{-\Phi}$
  - transforms by Möbius transformations under $SL(2,\mathbb{Z})$

$$\rho \to \frac{a\rho + b}{c\rho + d} \qquad a, b, c, d, \in \mathbb{Z}, \quad ad - bc = 1$$

  - Flux fields F_3^R, F_3^{NS} transform linearly; F_5 is invariant

- **Effective interactions from four-graviton amplitude in Type IIB**

$$\int \sqrt{G_E}\left(\mathcal{E}_0(\rho)\mathcal{R}_E^4 + \mathcal{E}_4(\rho)D_E^4\mathcal{R}_E^4 + \mathcal{E}_6(\rho)D_E^6\mathcal{R}_E^4 + \mathcal{E}_8(\rho)D_E^8\mathcal{R}_E^4 + \cdots\right)$$

  - For each p the real-valued function $\mathcal{E}_p(\rho)$ is $SL(2,\mathbb{Z})$-invariant

$$\mathcal{E}_p\left(\frac{a\rho + b}{c\rho + d}\right) = \mathcal{E}_p(\rho)$$

  - namely it is a *real-analytic modular function*
    (not to be confused with meromorphic modular functions)

Real-analytic Eisenstein series

- **A famous family of real-analytic modular functions**

  – For $\mathrm{Re}(s) > 1$ one defines E_s by Kronecker-Eisenstein sums

$$E_s(\rho) = \sum_{m,n \in \mathbb{Z}}{}' \frac{\rho_2^s}{\pi^s |m + \rho n|^{2s}} \qquad \rho = \rho_1 + i\rho_2, \; \rho_1, \rho_2 \in \mathbb{R}$$

  – They are $SL(2,\mathbb{Z})$-invariant and eigenfunctions of the Laplacian

$$\Delta E_s(\rho) = s(1-s)E_s \qquad \Delta = 4\rho_2^2 \partial_\rho \partial_{\bar\rho}$$

  – Their asymptotic expansion for $\rho_2 \to \infty$ = weak string coupling

$$E_s(\rho) = 2\zeta(2s)\frac{\rho_2^s}{\pi^s} + \frac{2\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\pi^{s-\frac{1}{2}}\rho_2^{s-1}} + \mathcal{O}(e^{-2\pi\rho_2})$$

Effective interactions and Eisenstein series

- **String perturbation theory calculations in string frame**
  - Convert Einstein metric $G_{E\mu\nu}$ to string metric $G_{\mu\nu} = e^{\Phi/2} G_{E\mu\nu}$

$$\sqrt{G_E}\, \mathcal{E}_{2k}(\rho) D_E^{2k} \mathcal{R}_E^4 = \mathbf{e}^{(\mathbf{k-1})\mathbf{\Phi}/\mathbf{2}} \sqrt{G} \mathcal{E}_{2k}(\rho) D^{2k} \mathcal{R}^4$$

  - Consider combinations involving Eisenstein series

$$\begin{aligned}
\sqrt{G_E}\, E_{\frac{3}{2}}(\rho) \mathcal{R}_E^4 &\approx e^{-2\Phi} \zeta(3) \mathcal{R}^4 + \frac{\pi^2}{3} \mathcal{R}^4 \\
\sqrt{G_E}\, E_{\frac{5}{2}}(\rho) D_E^4 \mathcal{R}_E^4 &\approx e^{-2\Phi} \zeta(5) D^4 \mathcal{R}^4 + \frac{2\pi^4}{135} e^{2\Phi} D^4 \mathcal{R}^4 \\
\sqrt{G_E}\, E_{\frac{3}{2}}(\rho)^2 D_E^6 \mathcal{R}_E^4 &\approx e^{-2\Phi} \zeta(3)^2 D^6 \mathcal{R}^4 + \frac{2\pi^2}{3} \zeta(3) D^6 \mathcal{R}^4 + \frac{\pi^4}{9} e^{-2\Phi} D^6 \mathcal{R}^4 \\
\sqrt{G_E}\, E_{\frac{7}{2}}(\rho) D_E^8 \mathcal{R}_E^4 &\approx e^{-2\Phi} \zeta(7) D^8 \mathcal{R}^4 + \frac{16\pi^6}{14175} e^{-4\Phi} D^8 \mathcal{R}^4
\end{aligned}$$

  - Compare with low energy expansion of tree-level

$$\frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2+t^2+u^2) + 2\zeta(3)^2 stu - \frac{1}{2}\zeta(7)(s^2+t^2+u^2)^2 + \cdots$$

$$\mathcal{R}^4 \qquad D^4\mathcal{R}^4 \qquad D^6\mathcal{R}^4 \qquad D^8\mathcal{R}^4$$

D-instantons, S-duality and supersymmetry

- **Space-time supersymmetry and S-duality**
  - D-instantons (Green, Gutperle, Vanhove 1996), space-time susy (Green, Sethi 1997)

$$\mathcal{E}_0(\rho) = E_{\frac{3}{2}}(\rho)$$

  - matches tree-level and genus-one results from string perturbation theory
  - Vanishing contribution from genus-two (ED, Gutperle, Phong 2005)

- **M-theory perturbation theory on torus** (Green, Kwon, Vanhove 1999; GV 2005)

$$\begin{aligned} \mathcal{E}_4(\rho) &= E_{\frac{5}{2}}(\rho) \\ (\Delta - 12)\mathcal{E}_6(\rho) &= E_{\frac{3}{2}}(\rho)^2 \end{aligned}$$

  - $\mathcal{E}_4$ matches genus two (ED, Gutperle, Phong 2005)
  - $\mathcal{E}_6$ matches genus-two (ED, Green, Pioline, R. Russo 2014)
    genus three (Gomez, Mafra 2015)

- **Non-renormalization theorems**: no perturbative corrections
  - for $\mathcal{E}_0$ for $h \geq 2$
  - for $\mathcal{E}_4$ for $h \geq 3$
  - for $\mathcal{E}_6$ for $h \geq 4$

Low energy expansion at genus one

- **Recall genus-one Type II four-graviton amplitude** $(\mathcal{M}_1 = \mathcal{H}_1/SL(2,\mathbb{Z}))$

$$\mathcal{A}^{(1)}(\varepsilon_i, \tilde{\varepsilon}_i, k_i) = \mathcal{R}^4 \int_{\mathcal{M}_1} \frac{d^2\tau}{(\text{Im}\,\tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau)$$

- **Expand the partial amplitude** $\mathcal{B}^{(1)}$ **for** $|s_{ij}| \ll 1$ **for fixed** τ

$$\mathcal{B}^{(1)}(s_{ij}|\tau) = \int_{\Sigma^4} \prod_{i=1}^{4} \frac{d^2 z_i}{\text{Im}\,\tau} \exp\Big(\sum_{i<j} s_{ij}\, G(z_i - z_j|\tau) \Big)$$

  – Scalar Green function $G(z|\tau)$ given by "Kronecker-Eisenstein" Fourier sum

$$G(z|\tau) = \sum_{m,n\in\mathbb{Z}}' \frac{\tau_2}{\pi} \frac{e^{2\pi i(m\beta - n\alpha)}}{|m+\tau n|^2} \qquad\qquad z = \alpha + \tau\beta,\ \alpha,\beta \in \mathbb{R}$$

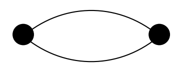
  – For fixed τ the Taylor expansion of $\mathcal{B}^{(1)}$ in s_{ij} converges for $|s_{ij}| < 1$

- **Graphical expansion of** $\mathcal{B}^{(1)}(s_{ij}|\tau) \Longrightarrow$ **Modular Graph Functions of** τ

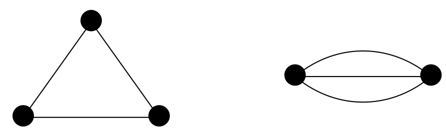
Modular graph functions

- **Graph in the expansion of $D^{2w}\mathcal{R}^4 \Longrightarrow$ Modular Function**

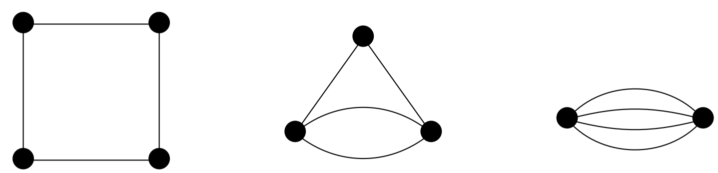
$D^4\mathcal{R}^4$



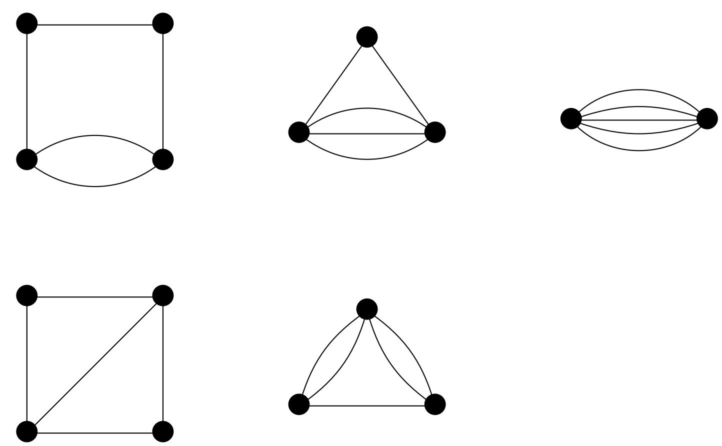
$D^6\mathcal{R}^4$



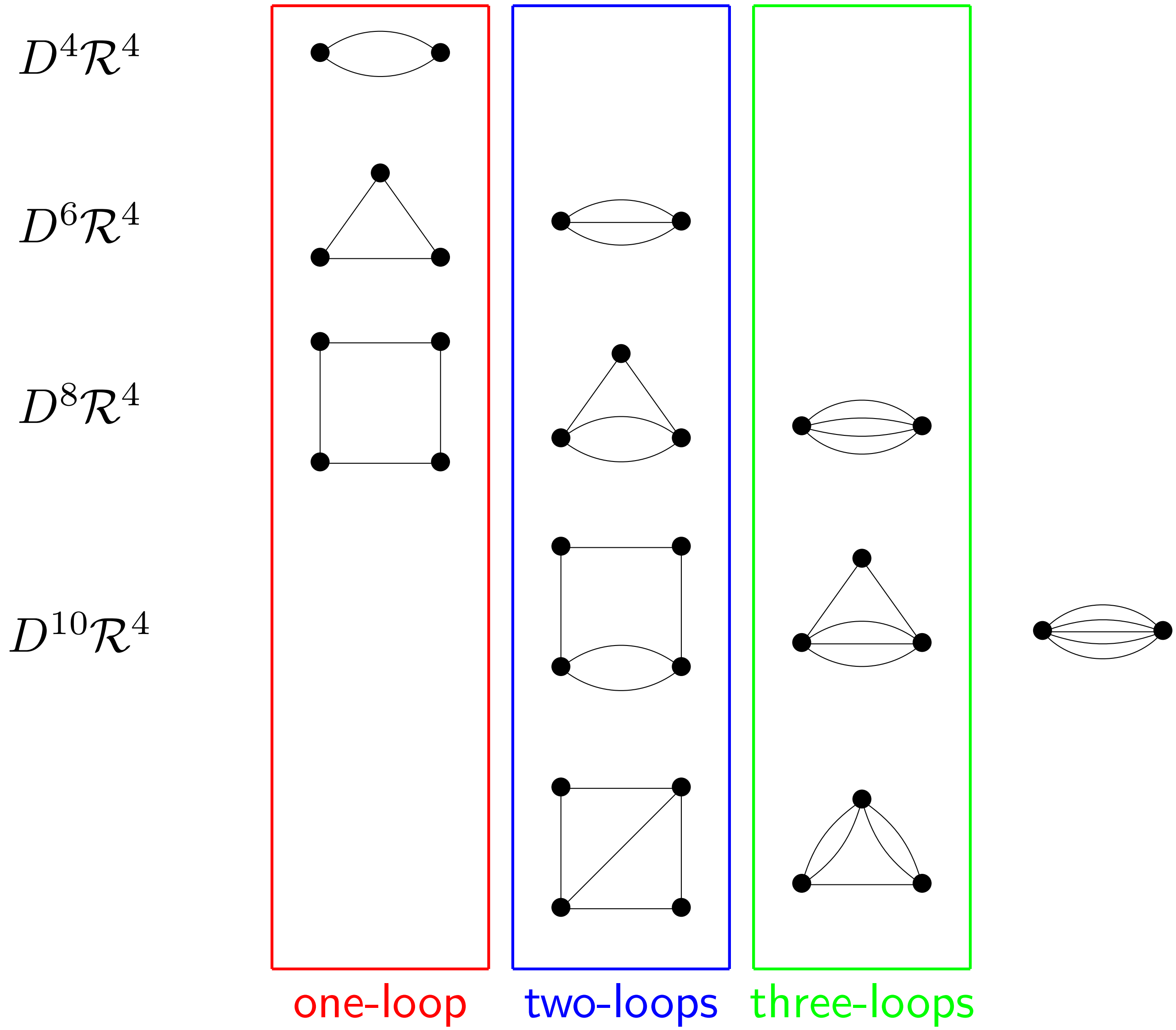
$D^8\mathcal{R}^4$



$D^{10}\mathcal{R}^4$



Modular graph functions



One-loop : Eisenstein series

- **One-loop worldsheet Feynman diagram with k bivalent vertices**

$$\prod_{i=1}^{k} \int_\Sigma \frac{d^2 z_i}{\tau_2} G(z_i - z_{i+1}|\tau) = \sum_{m,n\in\mathbb{Z}}{}' \frac{\tau_2^k}{\pi^k |m+n\tau|^{2s}} = E_k(\tau)$$

  – Our old friend: non-holomorphic Eisenstein series for integer index k

- **Recall properties of $E_s(\tau)$**

  – absolutely convergent for $\mathrm{Re}(s) > 1$; analytically continue to $s \in \mathbb{C}$
  – reflection relation $\Gamma(s)E_s(\tau) = \Gamma(1-s)E_{1-s}(\tau)$
  – satisfies a Laplace-eigenvalue equation on $\mathcal{H}_1$

$$\Big(\Delta - s(s-1)\Big)E_s(\tau) = 0 \qquad\qquad \Delta = 4\tau_2^2 \partial_\tau \partial_{\bar\tau}$$

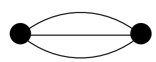
  – modular invariant $E_s(\frac{a\tau+b}{c\tau+d}) = E_s(\tau)$ under $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

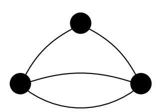
Two-loops : modular graph functions

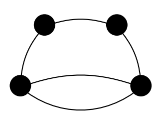
- **Feynman diagrams evaluate to the modular functions**

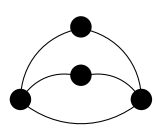
$$C_{a_1,a_2,a_3}(\tau) = \sum_{\substack{m_r,n_r\in\mathbb{Z},\\ r=1,2,3}}{}' \, \delta\left(\sum_{r=1}^3 m_r\right)\left(\sum_{r=1}^3 n_r\right)\prod_{r=1}^3\left(\frac{\tau_2}{\pi|m_r+n_r\tau|^2}\right)^{a_r}$$

– contribute to $D^{2w}\mathcal{R}^4$ with the *weight* given by $w = a_1 + a_2 + a_3$
– satisfy (inhomogeneous) Laplace-eigenvalue equations

$w=3$ $\quad C_{1,1,1} =$  $\quad (\Delta - 0)C_{1,1,1} = 6E_3$

$w=4$ $\quad C_{2,1,1} =$  $\quad (\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2$

$w=5$ $\quad C_{3,1,1} =$  $\quad (\Delta - 6)C_{3,1,1} = 3C_{2,2,1} + 16E_5 - 4E_2E_3$

$w=5$ $\quad C_{2,2,1} =$  $\quad (\Delta - 0)C_{2,2,1} = 8E_5$

– Note that eigenvalues are of the form $s(s-1)$ for $s = 1, 2, 3$;

Structure Theorem

- $C_{a,b,c}(\tau)$ **are linear combinations of** $\mathfrak{C}_{w;s;\mathfrak{p}}(\tau)$ **satisfying** (ED, Green, Vanhove 2015)

$$\big(\Delta - s(s-1)\big)\mathfrak{C}_{w;s;\mathfrak{p}} = \mathfrak{F}_{w;s;\mathfrak{p}}\big(E_{s'}\big)$$

  - $\mathfrak{C}_{w;s;\mathfrak{p}}$ and $\mathfrak{F}_{w;s;\mathfrak{p}}$ of weight $w = a+b+c$ (with $E_{s'}$ assigned weight s');
  - $\mathfrak{F}_{w;s;\mathfrak{p}}$ is a polynomial of total degree 2 in $E_{s'}$ with $2 \leq s' \leq w$;

$$s = w - 2\mathfrak{m} \qquad \mathfrak{m} = 1, \cdots, \left[\frac{w-1}{2}\right] \qquad \mathfrak{p} = 0, \cdots, \left[\frac{s-1}{3}\right]$$

- **Examples at low weight**

$w=3$	$s=1$	$0^{(1)}$
$w=4$	$s=2$	$2^{(1)}$
$w=5$	$s=1,3$	$0^{(1)} \oplus 6^{(1)}$
$w=6$	$s=2,4$	$2^{(1)} \oplus 12^{(2)}$
$w=7$	$s=1,3,5$	$0^{(1)} \oplus 6^{(1)} \oplus 20^{(2)}$

- **System of differential relations to all loop orders** (ED, Green, Kaidi, Vanhove 2016)
- **Modular generalizations of polylogarithms & multiple zeta values**
  (ED, Green, Vanhove 2015; Francis Brown 2017)

Type IIB effective interactions at genus-two

- **Recall Type II four-graviton amplitude at genus 2,**

$$\mathcal{A}^{(2)}(\varepsilon_i, k_i) = \mathcal{R}^4 \int_{\mathcal{M}_2} d\mu_2 \, \mathcal{B}^{(2)}(s_{ij}|\Omega)$$

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = \int_{\Sigma^4} \mathcal{Y} \wedge \bar{\mathcal{Y}} \exp \sum_{i<j} s_{ij} \, G(z_i, z_j)$$

  – $\mathcal{Y} = (s-t)\Delta(z_1, z_3) \wedge \Delta(z_4, z_2) + 2$ permutations;
  – $\Delta(z_i, z_j)$ is a holomorphic form independent of s, t, u.

- **Contributions to local effective interactions,**
  – $\mathcal{R}^4$: zero, since $\mathcal{Y}$ vanishes for $s = t = u = 0$;
  – $D^4\mathcal{R}^4$: non-zero, $\mathcal{B}^{(2)}$ constant on $\mathcal{M}_2$;
  – $D^6\mathcal{R}^4$: non-zero, one power of G brought down in integral over Σ^4;

$$\mathcal{B}^{(2)}(s_{ij}|\Omega) = 32(s^2 + t^2 + u^2) + 192 \, stu \, \varphi(\Omega) + \mathcal{O}(s^4, \cdots)$$

  – $\varphi(\Omega)$ coincides with the Kawazumi-Zhang invariant.

The Zhang-Kawazumi invariant for genus-two

- **The ZK-invariant is given as follows**

$$8\varphi(\Omega) = \sum_{I,J,K,L} \left(Y^{-1}_{IJ} Y^{-1}_{KL} - 2 Y^{-1}_{IL} Y^{-1}_{JK} \right) \int_{\Sigma^2} G(x,y) \omega_I(x) \overline{\omega_J(x)} \omega_K(y) \overline{\omega_L(y)}$$

  – equivalent to definition via Arakelov geometry (Zhang 2007, Kawazumi 2008)

- **Coefficient of genus-two $D^6\mathcal{R}^4$ interaction involves** $\int_{\mathcal{M}_2} d\mu_2 \, \varphi(\Omega)$

  – Direct evaluation appeared completely out of reach ... until ...

The Zhang-Kawazumi invariant for genus-two

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- **Theorem** (ED, Green, Pioline, R. Russo 2014)

$$(\Delta - 5)\varphi = -2\pi \delta_{SN}$$

  – Δ is the Laplace-Beltrami operator on $\mathcal{M}_2$ with Siegel metric ds^2_2;
  – δ_{SN} has support on separating node (into two genus-one surfaces)
  – The integral over $\mathcal{M}_2$ reduces to an integral over $\partial\mathcal{M}_2$

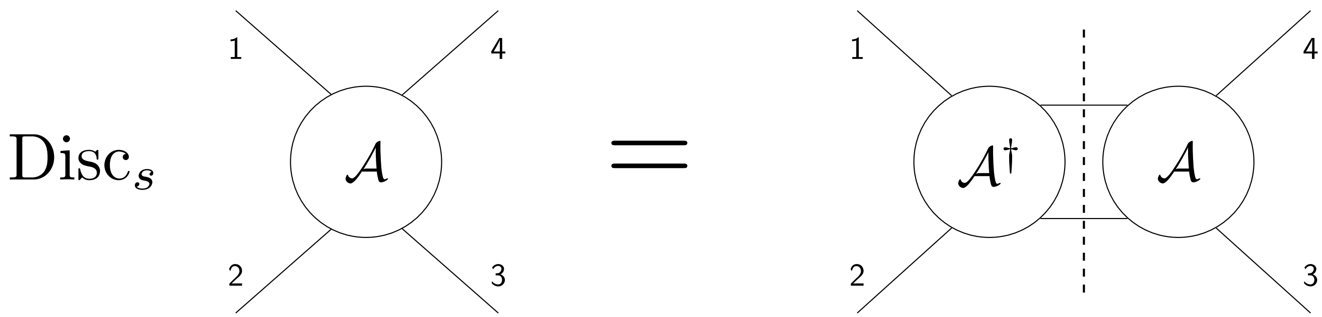
$$\int_{\mathcal{M}_2} d\mu_2 \, \varphi = \frac{1}{5} \int_{\mathcal{M}_2} d\mu_2 \Big(\Delta\varphi + 2\pi\delta_{SN} \Big) = \frac{2\pi^3}{45}$$

  – Exact agreement with predictions from S-duality and supersymmetry

Non-analytic contributions at low energy

- **Non-analytic parts of the amplitudes at low energy**
  - arise from boundary of moduli space contribution to the integral over $\mathcal{B}$
  - dominant contribution at low energy is from supergravity
  - plus string corrections

- **Look at two-particle unitarity cut in the s-channel**



$$i \, \mathrm{Disc}_s \, \mathcal{A}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(p_1,p_2,p_3,p_4) \; =$$

$$\int \frac{d^{10}k}{(2\pi)^{10}} \delta(k^2)\delta((q-k)^2)\mathcal{A}_{\varepsilon_1,\varepsilon_2,\varepsilon_r,\varepsilon_s}(p_1,p_2,-k,k-q)\mathcal{A}_{\varepsilon_r,\varepsilon_s,\varepsilon_3,\varepsilon_4}(k,q-k,p_3,p_4)$$

Non-analytic part of the genus-one amplitude

- **Obtain the genus-one discontinuity from tree-level**
  - Use the fact that the kinematic factor is the same at all genera h

$$\mathcal{A}^{(h)}_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(p_1,p_2,p_3,p_4) = \mathcal{R}^4_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(p_1,p_2,p_3,p_4)\mathcal{A}^{(h)}_{\rm red}(s,t,u)$$

  - and satisfies the recursive formula (Bern, Dixon, Dunbar, Perelstein, Rozowsky 1998)

$$\sum_{\varepsilon_r,\varepsilon_s} \mathcal{R}^4_{\varepsilon_1,\varepsilon_2,\varepsilon_r,\varepsilon_s}(p_1,p_2,-k,k-q)\mathcal{R}^4_{\varepsilon_r,\varepsilon_s,\varepsilon_3,\varepsilon_4}(k,q-k,p_3,p_4) = s^4\,\mathcal{R}^4_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(p_1,p_2,p_3,p_4)$$

  - to obtain an effective discontinuity formula

$$i\,{\rm Disc}_s\,\mathcal{A}^{(1)}_{\rm red}(s,t,u) = \int \frac{d^{10}k}{(2\pi)^{10}}\delta(k^2)\delta((q-k)^2)\mathcal{A}^{(0)}_{\rm red}(s,t',u')\mathcal{A}^{(0)}_{\rm red}(s,t'',u'')$$

  - where $t' = -(p_1-k)^2,\ u' = -(p_2-k)^4,\ t'' = -(p_4-k)^2,\ u'' = -(p_3-k)^2$

$$\mathcal{A}^{(0)}_{\rm red}(s,t,u) = \frac{1}{stu} + 2\zeta(3) + \zeta(5)(s^2+t^2+u^2) + \cdots$$

- **Substitution into s-channel unitarity relation gives** (by power-counting)

$$\begin{aligned}{\rm Disc}_s\,\mathcal{A}^{(1)}_{\rm red}(s,t,u) &= \#s + \#\zeta(3)s^4 + \#\zeta(5)s^6 + \cdots \\ \mathcal{A}^{(1)}_{\rm red}(s,t,u) &= \#s\ln(-s) + \#\zeta(3)s^4\ln(-s) + \#\zeta(5)s^6\ln(-s) + \cdots\end{aligned}$$

Absence of non-analytic contributions

- **Discontinuity relation gives non-analytic contributions**
  - At genus-one use previously obtained result

$$\mathcal{A}^{(1)}_{\rm red}(s,t,u) = \# s \ln(-s) + \#\zeta(3) s^4 \ln(-s) + \#\zeta(5) s^6 \ln(-s) + \cdots$$

  - Effective interaction $D^2\mathcal{R}^4$ vanishes by $s+t+u=0$
  - Genus-one $\mathcal{R}^4$, $D^4\mathcal{R}^4$, $D^6\mathcal{R}^4$, $D^{10}\mathcal{R}^4$ effective interactions are completely determined by the analytic part of the amplitude
  - Local effective interaction $D^8\mathcal{R}^4$ can be fixed only after non-analytic part has been properly normalized

Non-analytic plus analytic parts from genus-one amplitude

- **Derivation of full genus-one $D^8\mathcal{R}^4$ from string theory amplitude**

  – non-analytic part arises from $\tau \to i\infty$: partition moduli space

$$\mathcal{M}_1 = \mathcal{M}_{1L} \cup \mathcal{M}_{1R} \qquad \mathcal{M}_{1L} = \{\tau \in \mathcal{M}_1, \, \mathrm{Im}(\tau) < L\}$$

$$\mathcal{M}_{1R} = \{\tau \in \mathcal{M}_1, \, \mathrm{Im}(\tau) > L\}$$

  – Full amplitude is a sum $\mathcal{A}^{(1)} = \mathcal{A}_L^{(1)} + \mathcal{A}_R^{(1)}$

$$\mathcal{A}_{L,R}^{(1)}(\varepsilon_i, \tilde\varepsilon_i, k_i) = \mathcal{R}^4 \int_{\mathcal{M}_{1L,R}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \mathcal{B}^{(1)}(s_{ij}|\tau)$$

  – Both $\mathcal{A}_L^{(1)}$, $\mathcal{A}_R^{(1)}$ depend on L, but sum is independent of L

  – $\mathcal{A}_L^{(1)}$ is analytic in s_{ij} but $\mathcal{A}_R^{(1)}$ exhibits non-analyticity at $s_{ij} = 0$

Explicit calculation for $D^8\mathcal{R}^4$

- **Since $\mathcal{A}_L^{(1)}$ is analytic in s_{ij}, evaluate using modular graph functions**

$$\mathcal{A}_L^{(1)} = \frac{2\pi\zeta(3)}{45}\mathcal{R}^4 \left(\ln L - \frac{1}{4} + \ln 2 + \frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)} \right) + \text{power-behaved in } L$$

- **For $L \gg 1$, approximate integrand of $\mathcal{A}_R^{(1)}$ by supergravity + corrections**

$$\begin{aligned} \mathcal{A}^{(1)}\Big|_{D^8\mathcal{R}^4} = \; & \frac{4\pi\zeta(3)}{45}\left(\frac{17}{5} - \frac{1}{4} + \ln 2 + \frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta'(3)}{\zeta(3)}\right)(s^4+t^4+u^4)\,\mathcal{R}^4 \\ & -\frac{4\zeta(3)}{45}\Big(s^4\ln(-2\pi s) + t^4\ln(-2\pi t) + u^4\ln(-2\pi u)\Big)\mathcal{R}^4 \end{aligned}$$

  - Note: no ambiguities, no infinities, no renormalization required !
  - Transcendentality ... (ED, Green, in progress)

- **Genus-two story ...**

  (ED, Green, Pioline 2017, 2018, and in progress)

Outlook

- **Some additional developments**
  - Clarification of super Riemann surfaces with R-punctures (Witten 2012)
  - There exists a super-period matrix for R-punctures (Witten; ED, Phong 2015)
  - New relations between open and closed string amplitudes (Schlotterer et al.)

- **Some outstanding issues**
  - Systematic structure of low energy effective interactions w/ Green, Pioline
    - ⋆ in terms of properties of modular graph functions
    - ⋆ calculation without requiring subtleties of supermoduli space
    - ⋆ UV divergences in supergravity and effective interactions
  - Ambi-twistor strings
  - string perturbation theory on curved spaces with RR flux, e.g. $AdS_5 \times S^5$