Lectures on Superstring Amplitudes

Part 1: Bosonic String

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Outline of lectures

• Lecture 1
  Bosonic strings and conformal field theory

• Lecture 2
  Superstring amplitudes

• Lecture 3
  Low energy effective interactions, modular graph functions
Strings

- **A string is a 1-dimensional object**
  - open string = topology of an interval;
  - closed string = topology of a circle;
  - physical size Planck length \( \ell_P \approx 10^{-33}\text{cm} \approx 10^{-19} \times \text{size of the proton} \).

- **Ultimate goal: unified theory of particle physics and gravity**
  - elementary particles correspond to strings and their excited states;
  - consistently with quantum mechanics and general relativity;
  - remarkably unique structure.

- **Immediate goal: relating string amplitudes and field theory amplitudes**
  - at distance scales larger than the Planck length (low energy)
    - a string effectively behaves as a point particle
  - string theory exhibits powerful structure of amplitudes
String Topology

- **Consistent interacting string theories**
  - only closed strings (Type IIA, B and heterotic)
  - closed and open strings (Type I)
  - Type II theories have open strings in the presence of D-branes

- **Strings live in a physical space-time** $M$
  - $M$ may be a manifold or an orbifold (with mild isolated singularities)
  - superstring theory predicts 10-dim
  - but space-time visible to us is 4-dim. $\Rightarrow$ requires “compactification”

- **Under time-evolution strings sweep out a 2-dim. surface**

  ![Diagram](image)

  - closed strings
  - time-evolution (freely propagating)
  - basic interaction (purely topological)
Perturbative String Amplitudes

- **Quantum probability** “scattering” amplitudes
  - = Feynman functional integral/sum over all surfaces with given boundary components for initial and final strings

- **Closed oriented string perturbation theory**
  - The only remaining topological characterization is the genus $h \geq 0$
    - probability amplitude includes sum over all genera
    - weighed by a factor $g_s^{2h-2}$ where $g_s$ is the “string coupling”

\[
g_s^{-2} + g_s^0 + g_s^2 + \cdots
\]

- genus $h = \text{number of “loops”}$
Structure of string amplitudes

- **Perturbative part of string amplitude decomposes into a sum over topologies**

\[ A_{\text{perturbative}} = \sum_{h=0}^{\infty} g_s^{2h-2} \times A^{(h)} \]

- \( A^{(h)} \) is the amplitude at genus \( h \)

- The perturbative expansion in \( g_s \) is asymptotic but not convergent (just as in field theory)

- **Non-perturbative part** (not considered here)

  * instantons \( \approx e^{-1/g_s^2} \)
  * D-branes contribute \( \approx e^{-1/g_s} \).
**String Data** (closed oriented bosonic strings)

- **Assume fixed space-time** $M$, with fixed metric $G$
  - Physical space-time has Minkowski signature metric $G$
  - Starting point for string theory is often a Riemannian metric (if needed to be analytically continued to Minkowski signature)

- **The 2-dimensional worldsheet** $\Sigma$ is mapped into space-time $M$
  - The space of all such maps $x : \Sigma \to M$ is denoted $\text{Map}(\Sigma)$.

- **Riemannian metric** $G$ *induces* a Riemannian metric $x^*(G)$ on $\Sigma$
  - Hence $\Sigma$ is a Riemann surface (i.e. complex manifold with holé transition functions)

- **Polyakov formulation invokes an independent metric**
  - Riemannian metric $g$ on $\Sigma$
  - Denote the $\infty$-dim. Riemannian manifold of such metrics by $\text{Met}(\Sigma)$
  - String amplitude at fixed genus $h$ obtained by weighed sum over $g, x$

$$A^{(h)} = \int_{\text{Met}(\Sigma)} Dg \int_{\text{Map}(\Sigma)} Dx \, e^{-I_G[x,g]}$$
The worldsheet action $I_G$ and the measure $Dx$

- **Basic Criteria**
  - Intrinsic = invariant under "reparametrizations" $\text{Diff}(\Sigma)$ of $\Sigma$
  - lead to a well-defined QFT (renormalizable)

- **e.g.** **Non-linear sigma model action** with Riemannian metric $G$

  $$I_G[x, g] = \frac{1}{\alpha'} \int_{\Sigma} d^2\xi \sqrt{g} g^{mn} \partial_m x^\mu \partial_n x^\nu G_{\mu\nu}(x)$$

  $m, n = 1, 2$  
  worldsheet indices
  $\mu, \nu = 1, \cdots, D$  
  space-time Einstein indices

- **The measure is governed by the $L^2$-norm**

  $$||\delta x||^2_G = \int_{\Sigma} d^2\xi \sqrt{g} \delta x^\mu \delta x^\nu G_{\mu\nu}(x)$$

  - manifestly intrinsic
  - renormalizable in a generalized sense (the metric $G$ is renormalized)
**Weyl(Σ)-invariance**

- **Weyl transformations:** \( g_{mn} \rightarrow e^{2\sigma} g_{mn} \) leaving \( x^\mu \) and \( G \) unchanged

- **The classical action** \( I_G \) is Weyl-invariant for any metric \( G \)
  - but the measure \( Dx \) is not Weyl-invariant
  - which gives rise to a “Weyl-anomaly”
    = symmetry of classical action not preserved by quantization

- **The action** \( I_G \) defines a conformal quantum field theory
  \[
e^{-W_G[g]} = \int_{\text{Map}(\Sigma)} Dx \ e^{-I_G[x,g]}\]
  - provided \( W_G \) is \( \text{Diff}(\Sigma) \)-invariant
  - obeys the following Ward identity under Weyl transformations
    \[
    \delta W_G[g] = \frac{c}{24} \int_{\Sigma} d^2 \xi \sqrt{g} R_g \delta \sigma
    \]
    = where \( R_g \) is the scalar curvature of the metric \( g \) on the surface \( \Sigma \)

- **The measure** \( Dg \) is not Weyl-invariant, but the combined amplitude
  - is Weyl invariant for central charge \( c = 26 = \dim(M) \)
  - later we shall see for the superstring \( D = 10 = \dim(M) \)
Conformal Field Theory

- **Stress tensor encodes response of field theory to change in metric**

\[ T^c_{mn} = \frac{\delta W_G[g]}{\sqrt{g} \delta g^{mn}} \]

- Diff(\Sigma)-invariance requires “a conserved stress tensor” \( \nabla^m T^c_{mn} = 0 \)
- Weyl anomaly requires \( g^{mn} T^c_{mn} = -\frac{c}{12} R_g \)

- **Traceless stress tensor** \( T_{mn} \) obtained by adding a local counter-term
  - In local complex coordinates \((z, \tilde{z})\) we have \(T_{z\tilde{z}} = T_{\tilde{z}z} = 0\) and
    \[ T_{zz} = T^c_{zz} + \frac{c}{6} \left( 2\partial_z \Gamma^z_{zz} - (\Gamma^z_{zz})^2 \right) \]
    \[ \Gamma^z_{zz} = \partial_z \ln g_{z\tilde{z}} \]
  - Successive derivatives of \(W\) in \(g_{mn}\) give correlators of \(T_{mn}\)
  - Their singular part is governed by the OPE and the Ward identities
\[
T_{zz} T_{ww} = \frac{c/2}{(z - w)^4} + \frac{2T_{ww}}{(z - w)^2} + \frac{\partial_w T_{ww}}{z - w} + \text{regular}
\]
  - The mode expansion \(T_{zz} = \sum_m z^{-2-m} L_m\) gives the Virasoro algebra
\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0}
\]
Negative norm states

- Consider flat Minkowski $M = \mathbb{R}^{26}$ with metric $\eta = \text{diag}(- + \cdots +)$
- Maps $x: \Sigma \to M$ satisfy Laplace equation $\partial_{\tilde{z}} \partial_{\tilde{z}} x^\mu = 0$ for $\mu = 1, \ldots, 26$
- Concentrate on holomorphic field
  $$\partial_{\tilde{z}} x^\mu = \sum_{m \in \mathbb{Z}} x_m^\mu \tilde{z}^{-m-1} \quad [x_m^\mu, x_n^\nu] = m \delta_{m+n,0} \eta^{\mu\nu} \quad (x_n^\mu)^\dagger = x_{-n}^\mu$$
- Similarly anti-holomorphic field $\partial_{\tilde{z}} x^\mu$ produces modes $\tilde{x}^\mu$

- Single string ground state $|0, k\rangle$ labeled by its momentum $k$ satisfies
  $$x_0^\mu |0, k\rangle = k^\mu |0, k\rangle \quad x_m^\mu |0, k\rangle = 0 \text{ for } m > 0$$
- Fock space (holo sector) generated by linear combinations of
  $$x_{m_1}^{\mu_1} \cdots x_{m_p}^{\mu_p} |0, k\rangle \quad m_1, \ldots, m_p < 0$$
- Lowest excited state $\varepsilon_\mu(k) x_{-1}^{\mu} |0, k\rangle$ has norm
  $$\| \varepsilon_\mu(k) x_{-1}^{\mu} |0, k\rangle \|^2 = \varepsilon_\mu(k) \varepsilon_\nu(k) \eta^{\mu\nu} \| |0, k\rangle \|^2$$
- Component $\varepsilon^{\mu} = \delta^{\mu,0}$ produces negative norm state (assuming $\| |0, k\rangle \|^2 > 0$)
  = inconsistent with quantum mechanical probability interpretation
Eliminating negative norm states – conformal symmetry

- Conformal symmetry guarantees the existence of Virasoro algebra

\[ [L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m(m^2 - 1)\delta_{m+n,0} \]

- for the bosonic string \( c = 26 \) and

\[
L_m = \sum_{n \in \mathbb{Z}} \frac{1}{2} x_{m-n} \cdot x_n \\
L_0 = \frac{1}{2} x_0^2 + \sum_{n \in \mathbb{N}} x_{-n} \cdot x_n
\]

- A state \( |\psi\rangle \) is “physical” if \( (L_0 - 1)|\psi\rangle = L_m|\psi\rangle = 0 \) for \( m \in \mathbb{N} \)
  - Eliminates all negative norm states;
  - Decouples all null states produced by gauge transformations;

- e.g. on states \( |\psi\rangle = \varepsilon(k) \cdot x_{-1} |0, k\rangle \)
  - \( L_1 \) constraint imposes \( k \cdot \varepsilon(k) = 0 \)
  - \( L_0 \) constraint imposes \( k^2 = 0 \)
  - \( L_m \) constraints are automatic for \( m \geq 2 \) for this particular state

- the state \( |0, k\rangle \) itself is a tachyon (to be absent in the superstring)

⇒ Negative norm and null states eliminated by conformal symmetry
Conformal symmetry in curved space-times

- **Condition for Weyl-invariance on the metric** $G$
  - Infinitesimal Weyl variation for arbitrary $G$ to one-loop order in $\alpha'$

$$\delta W_G[g] = \int_\Sigma d^2\xi \sqrt{g} g^{mn} \partial_m x^\mu \partial_n x^\nu R_{\mu\nu}(x) \delta \sigma + \cdots + \mathcal{O}(\alpha')$$

  where $R_{\mu\nu}$ is the Ricci tensor of the metric $G_{\mu\nu}$

  - Thus, to leading order in $\alpha'$ conformal invariance requires $R_{\mu\nu} = 0$
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Vertex operators

• Small fluctuations in the metric are gravitons
  – A string couples to $N$ gravitons in flat space by slightly perturbing the metric
    \[
    G_{\mu\nu}(x) = \eta_{\mu\nu} + \sum_{i=1}^{N} \varepsilon_{i\mu\nu}(k_i) e^{ik_i\mu x^\mu} + \mathcal{O}(\varepsilon^2)
    \]
  – conformal invariance requires $G$ to satisfy the linearized Einstein equations
    \[
    k_i^2 = 0 \quad k_i^\mu \varepsilon_{i\mu\nu}(k_i) = 0 \quad \text{for } i = 1, \ldots, n
    \]

• Vertex operator formulation is obtained by expanding in powers of $\varepsilon_i$

\[
\mathcal{A} = \sum_{h=0}^{\infty} g_s^{2h-2} \int_{\text{Met}(\Sigma)} \int_{\text{Map}(\Sigma)} Dg \int Dx \ V_1[x, g] \cdots V_N[x, g] e^{-I_\eta[x, g]}
\]
  – where the vertex operator for an on-shell physical graviton is given by
    \[
    V_i[x, g] = \varepsilon_{i\mu\nu}(k_i) \int_{\Sigma} d^2\xi \sqrt{g} g^{mn} \partial_m x^\mu \partial_n x^\nu e^{ik_i\mu x^\mu}
    \]
  – On-shell conditions $k_i^2 = k_i \cdot \varepsilon_i = 0$ guarantee conformal invariance
Diff(Σ) × Weyl(Σ) and Moduli space

• Fix topology of Σ
  – Diff(Σ) re-parametrizes ξ^m on Σ by vector field δξ^m = −δυ^m
  \[ δg_{mn} = \nabla_m δυ_n + \nabla_n δυ_m \]
  – Weyl (Σ) \[ δg_{mn} = 2δσ g_{mn} \] with δσ an arbitrary real function of Σ

• Orbits of Diff(Σ) × Weyl(Σ) acting on the space Met(Σ)

\[ \text{Met}(Σ)/\text{Diff}(Σ) \times \text{Weyl}(Σ) = \mathcal{M}_h \]

• Moduli space \( \mathcal{M}_h \) of compact Riemann surfaces of genus \( h \) (no boundaries)
  = space of conformal structures (= space of complex structures)

\[ \text{dim}_\mathbb{C} \mathcal{M}_h = \begin{cases} 
0 & h = 0 \\
1 & h = 1 \\
3h - 3 & h \geq 2 
\end{cases} \]
Some trivial moduli spaces

- **Given an infinitesimal** $\delta g_{mn}$ **can one solve for** $\delta \sigma$ **and** $\delta v_m$ **?**
  \[
  \delta g_{mn} = 2\delta \sigma g_{mn} + \nabla_m \delta v_n + \nabla_n \delta v_m
  \]
  - Eliminate the trace part by choosing $\delta \sigma = g_{mn} \delta g_{mn} + \nabla_m \delta v^m$
  - In local complex coordinates $(z, \tilde{z})$, remaining eqs for traceless part
    \[
    \delta g_{zz} = \nabla_z v_z \quad \delta g_{\tilde{z}\tilde{z}} = \nabla_{\tilde{z}} v_{\tilde{z}}
    \]
  - Integrability automatic since $\nabla_z$ and $\nabla_{\tilde{z}}$ act on different functions
    $\Rightarrow$ locally, or in any simply connected set, you can always solve

- **The sphere** $S^2$ **has no moduli (compact)**
  - Its stereographic projection onto $\mathbb{C}$ admits a globally conformally flat metric
    \[
    ds^2 = \frac{|dz|^2}{(1 + |z|^2)^2}
    \]

- **The Poincaré upper half plane** $\mathcal{H}$ **has no moduli (non-compact)**
  \[
  ds^2 = \frac{|dz|^2}{(\text{Im } z)^2} \quad \text{Im } z > 0
  \]
Moduli deformations of the torus

- The torus may be viewed as the product of two circles $\mathcal{A}$ and $\mathcal{B}$
  - The ratio of their lengths and relative angle provide two real moduli
  - equivalently represented by parallelogram in $\mathbb{C}$ with sides pairwise identified

- The complex number $\tau$ contains the information of relative lengths and angle

- Constant metric deformations equivalently provide a complex modulus
  - translation invariance on the circles induces translation invariance on the torus
  - by translation invariance, metric is constant on $\Sigma$
  - constant trace-part of $\delta g_{mn}$ eliminated by constant $\sigma$
  - but constant $\delta g_{zz} = \partial_z v_z$ has no periodic solutions $v_z$
    $\Rightarrow$ constant $\delta g_{zz}$ provides the deformation of the complex modulus of the torus.
Moduli space of the torus

• Oriented Riemann surfaces: cycles \( \mathcal{A} \) and \( \mathcal{B} \) ordered
  - equivalently choose orientation \( \tau \in \mathcal{H}_1 = \{ \tau \in \mathbb{C}, \text{Im}(\tau) > 0 \} \)

• Space of inequivalent tori = space of inequivalent lattices \( \Lambda_\tau = \mathbb{Z} \oplus \tau \mathbb{Z} \)
  - but different values of \( \tau \) may give the same lattice
    \[ \begin{align*}
    \omega_1' &= a \omega_1 + b \omega_2 \\
    \omega_2' &= c \omega_1 + d \omega_2 \\
    \tau &= \omega_1/\omega_2 \\
    \tau' &= (a \tau + b)/(c \tau + d)
    \end{align*} \]

  - identical lattices requires \( \Lambda_{\tau'} \subset \Lambda_\tau \) and \( \Lambda_\tau \subset \Lambda_{\tau'} \)
  - so that \( a, b, c, d \in \mathbb{Z} \) and \( ad - bc = 1 \) and \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \)
  - generated by \( \tau \to \tau + 1 \) and \( \tau \to -\tau^{-1} \)

• Moduli space of tori = space of inequivalent lattices = \( \mathcal{H}_1/SL(2, \mathbb{Z}) \)
  - standard fundamental domain

\[
\mathcal{H}_1/SL(2, \mathbb{Z}) \equiv \left\{ \tau \in \mathcal{H}_1, |\tau| \geq 1, |\text{Re}(\tau)| \leq \frac{1}{2} \right\}
\]
Decomposing the measure $Dg$

- At any point $g \in \text{Met}(\Sigma)$ the measure $Dg$ factors

$$Dg = Z_g \times D\sigma \times Dv \times d\mu_{\mathcal{M}_h}$$

  Jacobian   Weyl   Diff$_0$   $\mathcal{M}_h$

- infinitesimal Weyl $\delta g_{mn} = \delta \sigma g_{mn}$
- infinitesimal Diff$_0$ $\delta g_{mn} = \nabla_m \delta v_n + \nabla_n \delta v_m$
- infinitesimal moduli deformations $\delta g_{mn}$

- Goal
  - compute $Z_g$
  - formulate $Z_g$ in terms of ghosts
  - omit volume factors $D\sigma Dv$ of the group $\text{Diff}^+(\Sigma) \rtimes \text{Weyl}(\Sigma)$

- To decompose $Dg$ we study tensor spaces (alias line bundles) on $\Sigma$
Tensor Spaces - Line Bundles on $\Sigma$

- A one-form $\phi = \phi_z dz + \phi_{\bar{z}} d\bar{z}$ on $\Sigma$ decomposes into $K \oplus \bar{K}$
  
  $K = \{\phi_z dz\}$ is the (space of sections of the) canonical bundle on $\Sigma$
  
  for $m \in \mathbb{Z}$ define $K^m = \{\phi_z ... z dz^m\}$ and $\bar{K}^m = \{\phi_{\bar{z}} ... \bar{z} d\bar{z}^m\} \approx K^{-m}$

- $L^2$ inner product for $\phi_1, \phi_2 \in K^m$
  
  $$(\phi_1, \phi_2) = \int_\Sigma d\bar{z}dz \sqrt{g} (g_{z\bar{z}})^{-m} \phi_1^* \phi_2$$

  The spaces $K^m$ and $K^n$ with $m \neq n$ are mutually orthogonal

- Covariant derivative on $\phi \in K^m$ decomposes $\nabla \phi = \nabla_{\bar{z}}^{(m)} \phi + \nabla_z^{(m)} \phi$
  
  $\nabla_{\bar{z}}^{(m)} : K^m \to K^{m+1}$ mutual adjoint operators $(\nabla_{\bar{z}}^{(m)})^\dagger = -\nabla_z^{(m+1)}$

  $\nabla_z^{(m)} : K^m \to K^{m-1}$ with $\nabla_{\bar{z}}^{(m)} = g_{z\bar{z}} \nabla_{\bar{z}}^{(m)}$

- Riemann-Roch and Vanishing Theorems
  
  $\dim_{\mathbb{C}} \text{Ker} \nabla_{\bar{z}}^{(m)} - \dim_{\mathbb{C}} \text{Ker} \nabla_{\bar{z}}^{(1-m)} = (2m - 1)(h - 1)$

  $\text{Ker} \nabla_{\bar{z}}^{(m)} = 0$ for $h \geq 2$ and $m \leq -1$ (no holó vector fields for $h \geq 2$)

  $\text{Ker} \nabla_{\bar{z}}^{(m)} = 0$ for $h = 0$ and $m \geq 1$ (no holó forms on the sphere)
Decomposing the tangent space to $\text{Met}(\Sigma)$

- **Orthogonal decomposition of** $T_g(\text{Met}(\Sigma))$
  
  $T_g(\text{Met}(\Sigma)) = \{\delta \sigma \ g_{zz}\} \oplus \{\delta g_{zz} = g_{zz} \delta \eta \bar{z}\} \oplus \{\delta g_{\bar{z}z} = g_{\bar{z}z} \delta \eta \bar{z}\}$

  $\delta \sigma \in \mathcal{K}^0$ \hspace{0.5cm} $\delta \eta \bar{z} \in \mathcal{K} \otimes \bar{\mathcal{K}}^{-1}$ \hspace{0.5cm} $\delta \eta \bar{z} \in \bar{\mathcal{K}} \otimes \mathcal{K}^{-1}$

- **$\text{Diff}_0$ acts by** $\delta \eta \bar{z} = \nabla^{(1)}_{\bar{z}} \delta v\bar{z}$
  
  - For $h \geq 1$, the range of the operator $\nabla^{(1)}_{\bar{z}}$ is NOT all of $\mathcal{K} \otimes \bar{\mathcal{K}}^{-1}$
  - The orthogonal complement of the range of $\nabla^{(1)}_{\bar{z}}$ is given by
    
    $$\text{Range} \nabla^{(1)}_{\bar{z}} \oplus \text{Ker}(\nabla^{(1)}_{\bar{z}})^\dagger = \mathcal{K} \otimes \bar{\mathcal{K}}^{-1} \approx \mathbb{K}^2$$

- **Holomorphic quadratic differentials** $\phi^j \in \text{Ker}\nabla^{(2)}_{\bar{z}} \approx \text{Ker}(\nabla^{(1)}_{\bar{z}})^\dagger$
  
  - Hence we may identify $\text{Ker}\nabla^{(2)}_{\bar{z}} = T^*_{(1,0)}(\mathcal{M}_h)$
  - One-forms $\delta m^j \in T^*_{(1,0)}(\mathcal{M}_h)$ given by linear forms on $\bar{\mathcal{K}} \otimes \mathcal{K}^{-1}$
    
    $$\delta m^j = (\delta \eta, \phi^j) = \int_{\Sigma} d\bar{z}dz \delta \eta \bar{z} \phi^j_{\bar{z}z}$$

  - Weyl-invariant pairing and vanishes on $\delta \eta \in \text{Range} \nabla_{\bar{z}}^{(1)}$
  - Riemann-Roch and Vanishing give $\dim_{\mathbb{C}} \mathcal{M}_h = 3h - 3$ for $h \geq 2$
Decomposing the measure $Dg$ (cont’d)

- Parametrize $\mathcal{M}_h$ by a slice in $\text{Met}(\Sigma)$ transverse to $\text{Weyl} \ltimes \text{Diff}_0$

  orbits of $\text{Diff}(\Sigma) \ltimes \text{Weyl}(\Sigma)$

  $m^j, \tilde{m}^j$ local coordinates on $\mathcal{M}_h$

- Carry out a change of integration variables

  $T_g(\text{Met}(\Sigma)) = \{\delta\sigma g_{\bar{z}z}\} \oplus \{\delta\eta_{\bar{z}}\} \oplus \{\delta\eta_{\bar{z}}\}$

  - Orthogonality implies that the measure factorizes $Dg = D\sigma D\eta D\bar{\eta}$
  - The change of variables is given by (repeated indices $j$ are summed)

  $\delta\eta_{\bar{z}} = \nabla_{\bar{z}}^{(-1)} \delta v^z + (\mu_j)_z \bar{z} \delta m^j$

  $\delta\eta_z = \nabla_z^{(1)} \delta v^\bar{z} + (\tilde{\mu}_j)_z \bar{z} \delta \tilde{m}^j$

  $(\mu_j)_{\bar{z}} = g^{\bar{z}\bar{z}} \frac{\partial g_{\bar{z}z}}{\partial m^j}$

  $(\tilde{\mu}_j)_{\bar{z}} = g^{\bar{z}\bar{z}} \frac{\partial g_{zz}}{\partial \tilde{m}^j}$
Ghosts

• Use standard rules to introduce ghosts for the determinant
  – gauge transformations \((\delta v^\bar{z}, \delta v^\bar{\bar{z}}) \rightarrow (c^\bar{z}, \tilde{c}^\bar{\bar{z}})\) Grassmann-odd ghosts
  – conjugate \((\delta \eta^\bar{z}, \delta \eta^\bar{\bar{z}}) \rightarrow (b_{zz}, \tilde{b}_{\bar{z}\bar{z}})\) Grassmann-odd anti-ghosts
  – extended ghost action

\[
\int \sum d^2 z \left[ b_{zz} \left( \partial_{\bar{z}} c^\bar{z} + \mu_j \delta m^j \right) + \tilde{b}_{\bar{z}\bar{z}} \left( \partial_{\bar{\bar{z}}} \tilde{c}^\bar{\bar{z}} + \tilde{\mu}_j \delta \tilde{m}^j \right) \right]
\]

  – Here \(\delta m^j, \delta \tilde{m}^j\) are differential one-forms which are Grassmann odd

• Integrating out \(\delta m^j, \delta \tilde{m}^j\) gives the standard ghost representation

\[
\int D(x^\mu, b, \tilde{b}, c, \tilde{c}) \mathcal{V}_1 \cdots \mathcal{V}_N e^{-IG-I_{gh}} \prod_j \delta(\langle b, \mu_j \rangle) \delta(\langle \tilde{b}, \tilde{\mu}_j \rangle) dm^j d\tilde{m}^j
\]

  – where \(I_{gh}\) is the standard ghost action

\[
I_{gh} = \int \sum d^2 z \left[ b_{zz} \partial_{\bar{z}} c^\bar{z} + \tilde{b}_{\bar{z}\bar{z}} \partial_{\bar{\bar{z}}} \tilde{c}^\bar{\bar{z}} \right]
\]

  – gauge fixed formulation has BRST invariance
  – for the sphere and the torus, quotient out by conformal automorphisms
Bosonic string has tachyon and no fermions: unphysical

- Warm-up: tree-level tachyon scattering amplitude
  - Tachyon vertex operator $V(k_i) = \int_{\Sigma} d^2z_i \sqrt{g(z_i)} : e^{ik \cdot x(z_i)} :$
  - Scalar Green function on the sphere with metric $|dz|^2/(1 + |z|^2)^2$

\[
\langle x^\mu(z) x^\nu(w) \rangle = \eta^{\mu\nu} G(z, w) \quad G(z, w) = -\ln \frac{|z - w|^2}{(1 + |z|^2)(1 + |w|^2)}
\]

- Sphere has no moduli, ghost and scalar partition functions are constant

\[
\langle \prod_{i=1}^{N} d^2z_i \sqrt{g(z_i)} : e^{ik \cdot x(z_i)} : \rangle = \prod_{i=1}^{N} d^2z_i \prod_{i<j} |z_i - z_j|^{\alpha' k_i \cdot k_j}
\]

  - Integrand invariant under $z_i \rightarrow (\alpha z_i + \beta)/(\gamma z_i + \delta)$ with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{C})$
  - Factor out volume of $SL(2, \mathbb{C})$ by fixing $z_N = \infty, z_{N-1} = 1, z_{N-2} = 0$

- The 4-tachyon amplitude with $s_{ij} = -\alpha'(k_i + k_j)^2/4$

\[
\frac{1}{g_s^2} \int_{\Sigma} d^2z |z|^{\alpha' k_1 \cdot k_2} |z - 1|^{\alpha' k_1 \cdot k_3} = \frac{\Gamma(-1 - s)\Gamma(1 - t)\Gamma(-1 - u)}{g_s^2 \Gamma(2 + s)\Gamma(2 + t)\Gamma(2 + u)}
\]

  - Tachyon poles at $s, t, u = -1$
Kawai-Lewellen-Tye (KLT) relations

- **Tree-level closed string amplitudes are bilinears in open string amplitudes**
  - Closed string amplitudes on the sphere, vertex operators in interior
  - Open string amplitude on upper half plane, vertex operators on boundary
  - Consider open and closed string 4-tachyon amplitudes

\[
A^{(0)}_{\text{open}}(s, t) = \int_0^1 d\xi |\xi|^{k_1 \cdot k_2} |1 - \xi|^{k_2 \cdot k_3}
\]
\[
A^{(0)}_{\text{closed}}(s, t, u) = \int_{S^2} d^2 z |z|^{2k_1 \cdot k_2} |1 - z|^{2k_2 \cdot k_3}
\]

- Parametrize \( z = \alpha + i\beta \) then \( z \)-integrand is analytic function of \( \beta \) with branch points at \( \beta = \pm i\alpha \) and \( \beta = \pm i(1 - \alpha) \)
- Deform \( \beta \)-contour from real to imaginary axis, but pick up phases

\[
\int_{S^2} d^2 z |z|^{2k_1 \cdot k_2} |1 - z|^{2k_2 \cdot k_3} = \sin(\pi k_2 \cdot k_3) \int_0^1 d\xi |\xi|^{k_1 \cdot k_2} |1 - \xi|^{k_2 \cdot k_3} \int_1^\infty d\eta |\eta|^{k_1 \cdot k_2} |1 - \eta|^{k_2 \cdot k_3}
\]

- Converting the second integral back to \( A^{(0)}_{\text{open}} \), we obtain the KLT relation

\[
A^{(0)}_{\text{closed}}(s, t, u) = \sin(\pi k_2 \cdot k_3) A^{(0)}_{\text{open}}(s, t) A^{(0)}_{\text{open}}(t, u)
\]

- Does the worldsheet secretly have a Minkowski signature structure?
- No generalization known to loop level