Amplituhedron meets Jeffrey-Kirwan Residue

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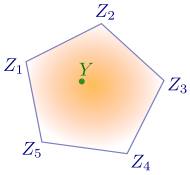
Amplitudes 2018 Summer School

UC Davis, QMAP, June 12, 2018

with Livia Ferro and Tomasz Łukowski, arXiv:1805.01301

Amplituhedron: Geometry and Volume

The **Amplituhedron** $\mathcal{A}$ - a recently discovered mathematical object: [Arkani-H., Trnka, '13]



- is a generalization of polytopes inside the Grassmannian
- has elements of the form $Y = C \cdot Z$

  + Positivity: C, Z have all ordered maximal minors positive

- is equipped with a **volume function** Ω such that

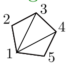
  Ω has logarithmic *singularities* at all *boundaries* of $\mathcal{A}$

- **Tree Amplitudes** in $\mathcal{N} = 4$ SYM can be extracted from Ω

How do we find Ω?

$\rightarrow$ *Geometrically*

Triangulate $\mathcal{A}$ & Sum over volumes of triangles



$\Omega = [123] + [134] + [145]$

Analytically $\leftarrow$

Evaluate a contour integral

$$\Omega = \int_\gamma \omega$$

different triangulations $\leftrightarrow$ different contours

Jeffrey-Kirwan Residue

The **Jeffrey-Kirwan Residue** is an operation on Differential Forms

[Jeffrey, Kirwan, '95]

Toy Example

$$\omega = \frac{dx_1 \wedge ... \wedge dx_r}{\beta_1(x)...\beta_n(x)}, \qquad \beta_i(x) = \beta_i \cdot x + \alpha_i$$

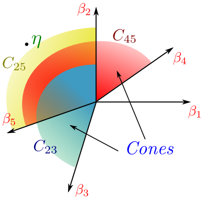
$$\omega = \frac{dx_1 \wedge dx_2}{\beta_1(x)...\beta_5(x)}$$

- For $B = \{\beta_i\}$ and fixed $\eta \in \mathbb{R}^r$, it is defined as

$$\mathrm{JKRes}^{B,\eta}\omega = \sum_{Cone \ni \eta} \mathrm{Res}_{Cone}\, \omega$$

multivariate residue

poles and sign prescribed by *Cones*



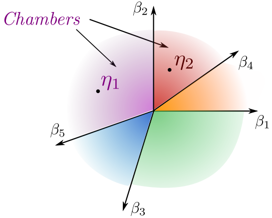
- Remarkable Property

JKRes is *independent* from the chamber

e.g. $\mathrm{JKRes}^{B,\eta_1} = \mathrm{Res}_{C_{25}} + \mathrm{Res}_{C_{45}} + \mathrm{Res}_{C_{23}}$

$||$

$\mathrm{JKRes}^{B,\eta_2} = \mathrm{Res}_{C_{45}} + \mathrm{Res}_{C_{12}} + \mathrm{Res}_{C_{42}}$



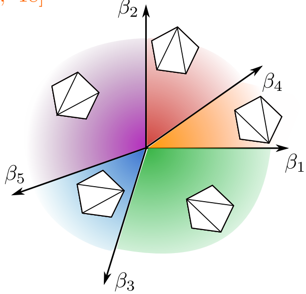
Amplituhedron meets Jeffrey-Kirwan Residue

For **Cyclic Polytopes** and **Conjugates** (*not Polytopes*!):

$$\Omega = \mathrm{JKRes}^{B,\eta}\omega$$

[Ferro, Łukowski, MP, '18]

Toy Amplituhedron

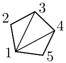


- *Positivity* of Amplituhedron $\leftrightarrow$ *configuration of Chambers*

  $\rightarrow$ *Geometrically*   *Analytically* $\leftarrow$

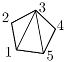
- each Chamber

  triangulation of $\mathcal{A}$ representation of Ω



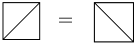
$$\mathrm{JKRes}^{B,\eta_1}\omega = [134] + [123] + [145]$$

$$\|$$



$$\mathrm{JKRes}^{B,\eta_2}\omega = [345] + [351] + [312]$$

- adjacent Chambers



$$[134] + [145] = [135] + [345]$$

Bistellar Flip   Global Residue Theorem

Amplituhedron meets Jeffrey-Kirwan Residue

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$$\Omega = \mathrm{JKRes}^{B,\eta}\omega$$

[Ferro, Łukowski, MP, '18]

- *Positivity* of Amplituhedron $\leftrightarrow$ *configuration of Chambers*

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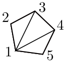
  triangulation of $\mathcal{A}$ | representation of Ω



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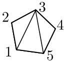
$$||$$



$$\mathrm{JKRes}^{B,\eta_2}\omega = [345] + [351] + [312]$$

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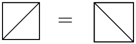
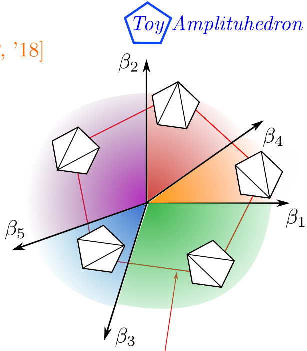


$$[134] + [145] = [135] + [345]$$

Bistellar Flip | Global Residue Theorem



Secondary Polytope $\Sigma(\mathcal{P})$:
vertices are triangulations of $\mathcal{P}$
e.g. $\Sigma(n\text{-gon})$ = *Associahedron*

$\mathcal{P} \rightarrow \mathcal{A}$

Secondary Amplituhedron

Conclusions and Outlook

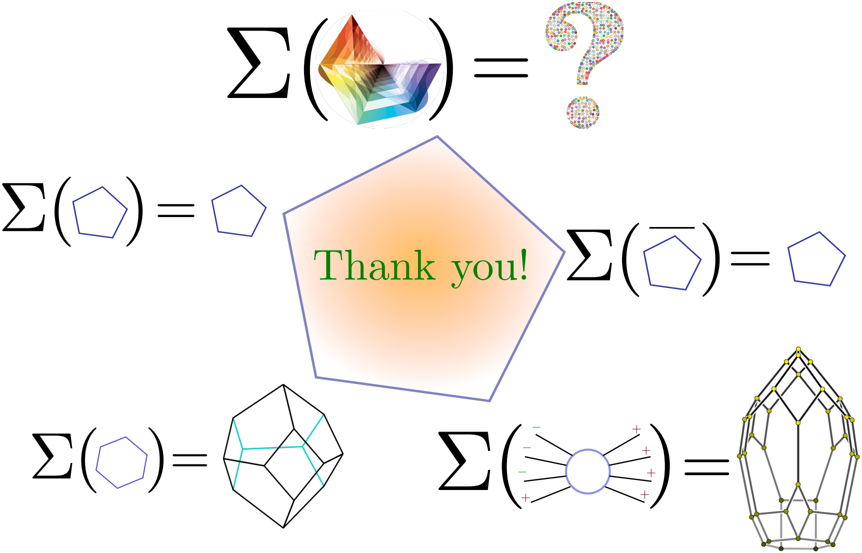
The Jeffrey-Kirwan Residue

- computes the volume of polytopes and their parity conjugates (*not polytopes*!)
- encodes all triangulations, in a triangulation-independent way
- points at the *Secondary Amplituhedron*, generalising Secondary Polytopes

Open Questions

- What is the *generalisation* of the Jeffrey-Kirwan Residue for all other Amplituhedra, i.e. other helicity sectors and loops?
- Can we find the Secondary Amplituhedron in all these cases?

  $\rightarrow$ many new representations of Scattering Amplitudes!


Thank you!