Triangulation of 2-loop MHV amplituhedron from sign flips

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Introduction

The amplituhedron

(N. Arkani-Hamed, J. Trnka 2013)

$$Y_a^I = C_{\alpha a} Z_\alpha^I$$

$C = (\vec{c}_1, \cdots, \vec{c}_n) \in G_+(k,n)$

$Z = (Z_1, \cdots, Z_n) \in M_+(k+m,n)$

$G_+(k,n)$:positive Grassmannian

M_+ :positive matrix

· N=4 SYM amplitudes = "Volume" of the amplituhedron.

· "Volume" $\cdots$ Canonical form Ω

The definition of canonical form Ω :

Ω has logarithmic singularities
on all the boundaries of amplituhedron.

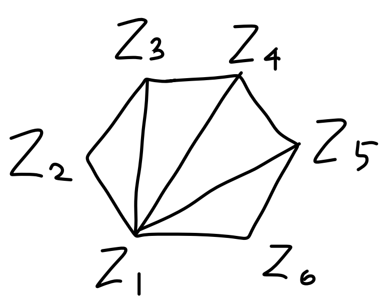
Triangulation

$$Y_a^I = C_{\alpha a} Z_\alpha^I \qquad \begin{array}{l} C \in G_+(k,n) \\ \dim\ C = k(n-k) \end{array} \longrightarrow \begin{array}{l} Y \in G_+(k,k+m) \\ \dim\ Y = mk \end{array}$$

$\longrightarrow$ highly redundant map

- Non-redundand map into $G_+(k,k+m)$ can only come from the $m \times k$ dimensional cells of $G_+(k,n)$.
- In the simple case, direct triangulation of the space is straightforward.

- Example: polygon



$$\Omega = \frac{\langle Y d^2 Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} + \frac{\langle Y d^2 Y\rangle\langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle} + \frac{\langle Y d^2 Y\rangle\langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle} + \frac{\langle Y d^2 Y\rangle\langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

- 1-loop MHV

$$\mathcal{L} = C \cdot Z \qquad \begin{array}{l} \mathcal{L} \in G_+(2,4) \\ C \in G_+(2,n) \end{array} \qquad \Omega_P = \sum_{i<j} \frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle AB(1ii+1)\cap(1jj+1)\rangle^2}{\langle AB1i\rangle\langle AB1i+1\rangle\langle ABii+1\rangle\langle AB1j\rangle\langle AB1j+1\rangle\langle ABjj+1\rangle}$$

- How to obtain the direct triangulation for more general case?

Introduction

- New picture of the amplituhdron: sign flip

(N. Arkani-Hamed, H.Thomas, J. Trnka 2017)

Y is in the m=2 k amplituhedron only if

$[Yii+1]>0$ and $\{[Y12],\cdots,[Y1n]\}$ has precisely k sign flips.

Y is in the m=4 loop amplituhedron only if

$$[(YAB)_\gamma ii+1]>0, [Yii+1jj+1]>0, [Y(AB)_\gamma(AB)_\rho]>0$$

$$\{[(YAB)_\gamma 12],\cdots,[(YAB)_\gamma 1n]\} \text{ has } (k+2) \text{ sign flip}$$

$$\{[Y1234],\cdots,[Y123n]\} \text{ has } k \text{ sign flip}$$

Amplituhedron from sign flip

- For m=1,2, the flip pattern give us a natural triangulation.

- In the case of m=2 k=2 and sign flip takes place in, i_1, i_2

$[Y1i_1]\ [Y1i_1+1]$

flip

$$Y_1 = Z_1 + x_1 Z_{i_1} + y_1 Z_{i_1+1}$$
$$Y_2 = -Z_1 + x_2 Z_{i_2} + y_2 Z_{i_2+1}$$

- From sign flip, $x_1, y_1, x_2, y_2 > 0$ and the canonical form for this flip pattern is

$$\Omega_{i_1 i_2} = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dx_2}{x_2}\frac{dy_2}{y_2}$$

- The canonical form for this amplituhedron is

$$\Omega = \sum_{\text{all flips}} \Omega_{ij}$$

- Example: 1-loop MHV integrand from sign flip

$$\Omega_P = \sum_{i<j} \frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle AB(1ii+1)\cap(1jj+1)\rangle^2}{\langle AB1i\rangle\langle AB1i+1\rangle\langle ABii+1\rangle\langle AB1j\rangle\langle AB1j+1\rangle\langle ABjj+1\rangle}$$

- This is the direct triangulation of 1-loop MHV integrand.

Amplituhedron from sign flip

- 2-loop case, $(AB)_1, (AB)_2$ are 1-loop amplituhedron

$$x_1, w_1, y_1, z_1 > 0, \; x_2, w_2, y_2, z_2 > 0$$

- And there are further constraint,

$$\langle (AB)_1 (AB)_2 \rangle > 0$$

- The region of these parameters are further bounded.

- Example: 4-pt

$$Z_A = Z_1 + x_1 Z_2 + w_1 Z_3, \quad Z_B = -Z_1 + y_1 Z_3 + z_1 Z_4,$$
$$Z_C = Z_1 + x_2 Z_2 + w_2 Z_3, \quad Z_D = -Z_1 + y_2 Z_3 + z_2 Z_4,$$

$$\langle ABCD \rangle = \langle 1234 \rangle \{ (x_1 - x_2)(y_1 z_2 - z_1 y_2) + (z_1 - z_2)(w_1 x_2 - x_1 w_2) \} > 0$$

- In the case of $(x_1 - x_2), (z_1 - z_2), (y_1 z_2 - y_2 z_1), (w_1 x_2 - w_2 x_1) > 0$

$$\begin{aligned} &0 < x_1, \quad 0 < x_2 < x_1 + \frac{(z_1 - z_2)(w_1 x_2 - x_1 w_2)}{y_1 z_2 - z_1 y_2}, \\ &0 < w_1, \quad 0 < w_2 < \frac{w_1 x_2}{x_1} \\ &0 < z_1, \quad 0 < z_2 < z_1, \\ &0 < y_1, \quad 0 < y_2 < \frac{y_1 z_2}{z_1} \end{aligned}$$

$$\Omega = \frac{1}{x_1} \left(\frac{1}{x_2} - \frac{1}{x_2 - x_1 - a} \right) \frac{1}{w_1 - \frac{x_1}{x_2} w_2} \frac{1}{w_2} \frac{1}{y_1 - \frac{z_1}{z_2} y_2} \frac{1}{y_2} \frac{1}{z_1 - z_2} \frac{1}{z_2}$$

Amplituhedron from sign flip

- 5pt case ··· there are 3×3=9 cells

$[AB12] \cdots [AB15] > 0$

$[AB12]$	$[AB13]$	$[AB14]$	$[AB15]$	
+	−	+	+	··· 23
+	+	−	+	··· 34
+	−	−	+	··· 24

$$\Omega = \Omega_{2323} + \Omega_{3434} + \Omega_{2424} + \Omega_{2324} + \Omega_{2334} + \Omega_{2434} + \Omega_{2423} + \Omega_{3423} + \Omega_{3424}$$

$$\Omega_{2324} = \frac{dx_1 dx_2 dw_1 dw_2 dy_1 dy_2 dz_1 dz_2}{x_1 x_2 w_1 w_2 y_1 y_2 z_1 z_2} \frac{1}{\langle ABCD \rangle} \{ \langle 1234 \rangle (x_1 w_2 y_2 + x_1 y_1 y_2 + x_2 w_1 z_1) \\ + \langle 1235 \rangle x_1 z_2 (w_2 + y_1) + z_1 z_2 (\langle 1345 \rangle w_1 + \langle 1245 \rangle x_1 + \langle 2345 \rangle x_2 w_1) \}$$

- The sum of these forms correspond to 5pt 2-loop integrand from BCFW.

Amplituhedron from sign flip

- n-pt case, there are $[\frac{1}{2}(n-3)(n-2)]^2$ cells from the way to chose i,j,k,l .

- There are 13 patterns that $i,j,k,l=2,3,\cdots,n-1, \quad i<j,k<l$

$$
\begin{aligned}
&i<k<l<j\cdots(1), \quad i<k<j<l\cdots(2), \quad i<j<k<l\cdots(3), \quad i=k<l<j\cdots(4),\\
&i=k<j=l\cdots(5), \quad i<k<j=l\cdots(6), \quad ,i<j=k<l\cdots(7), i=k<j<l\cdots(8),\\
&k<i=l<j\cdots(9), \quad k<i<l<j\cdots(10), \quad ,k<i<j=l\cdots(11), \quad ,k<i<j<l\cdots(12),\\
&k<l<i<j\cdots(13).
\end{aligned}
$$

$$\Omega = \sum_{\substack{i,j,k,l=2,3,\cdots,n-1\\ i<j,k<l}} \{\Omega^1_{ijkl} + \Omega^2_{ijkl} + \cdots + \Omega^{13}_{ijkl}\}$$

$$\Omega^1_{ijkl} = \frac{\langle ABjj+1\rangle\langle (AB)_i(CD)_j(CD)_l1\rangle - \langle ABii+1\rangle\langle (AB)_j(CD)_k(CD)_l1\rangle + \langle (AB)_i(AB)_j(CD)_k(CD)_l\rangle}{\langle ABii+1\rangle\langle ABjj+1\rangle\langle CDkk+1\rangle\langle CDll+1\rangle}.$$

$$
\begin{aligned}
\langle (AB)_l ijk\rangle &\equiv \langle AB(1ll+1)\cap(ikj)\rangle\\
\langle (CD)_k(AB)_l ij\rangle &\equiv \langle CD1k+1\rangle\langle (AB)_l ijk\rangle - \langle CD1k\rangle\langle (AB)_l ijk+1\rangle\\
\langle (EF)_j(CD)_k(AB)_l i\rangle &\equiv \langle EF1j+1\rangle\langle (CD)_k(AB)_l ij\rangle - \langle EF1j\rangle\langle (CD)_k(AB)_l ij+1\rangle\\
\langle (GH)_i(EF)_j(CD)_k(AB)_l\rangle &\equiv \langle GH1i+1\rangle\langle (EF)_j(CD)_k(AB)_l i\rangle - \langle GH1i\rangle\langle (EF)_j(CD)_k(AB)_l i+1\rangle.
\end{aligned}
$$

- This is the triangulation of n-pt 2-loop MHV integrand from sign flip.

- We checked that the sum of these forms correspond to the 2-loop integrand from BCFW up to 15-pt numerically.

- What is the structure of these cells?