Topological Entanglement Entropy in Euclidean AdS_3

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Topological Entanglement Entropy (TEE)

- For local gapped Hamiltonian, EE is area law in general.
- In 2+1d TQFT, an extra universal constant:
  [M. Levin, X.-G. Wen, PRL 2006; A. Kitaev, J. Preskill, PRL 2006]

$$S = \alpha L - \gamma \qquad \gamma = \log D$$

- For $\widehat{SU(2)}_k$ Chern-Simons, $1/D = \sqrt{\frac{2}{k+2}} \sin \frac{\pi}{k+2}$

  Used to compute TEE when spatial slice is sphere.

  [S. Dong, E. Fradkin, R. G. Leigh, S. Nowling, JHEP 2008]

3d Pure Gravity is TQFT

- No gravitational wave in 3d
- Einstein-Hilbert action $S_{\rm EH} \propto \int d^3x\sqrt{g}(R+2/l^2) + S_{\rm GH} + S_c$

$$= \int_{\mathcal{M}} d^3x\, e\left(R(e,\omega)+2/l^2\right) + \int_{\partial\mathcal{M}} e^a\omega^a$$

- [A. Achucarro, P. K. Townsend, PLB 1987; E.Witten, NPB 1988 & arXiv: 0706.3359 [hep-th]]

spin connection: ω — vielbein: e

$$A = \omega + e/l \qquad \bar{A} = \omega - e/l$$

- Chern-Simons action $S_{\rm CS}[A] = \int_M \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right)$

$$S_{\rm EH} \propto S_{\rm CS}[A] + S_{\rm CS}[\bar{A}]$$

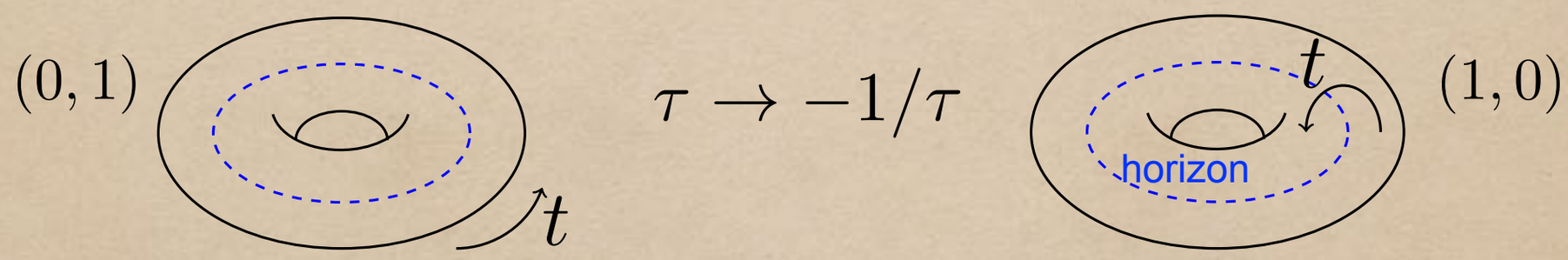
- Focus on AdS_3: gauge group $SO(3,1)$, no well-defined S-matrix

Euclidean AdS3

- Genus 1 classical solutions classified by integers (c, d)
- BTZ black hole [M. Bañados, C. Teitelboim, J. Zanelli, PRL 1992]

$$ds^2 = \frac{l^2}{\sin^2\chi}\left(\frac{dR^2}{R^2} + \cos^2\chi d\theta^2 + d\chi^2\right) \qquad (R,\theta,\chi) \sim \left(Re^{2\pi r_+/l}, \theta + \frac{2\pi|r_-|}{l}, \chi\right)$$

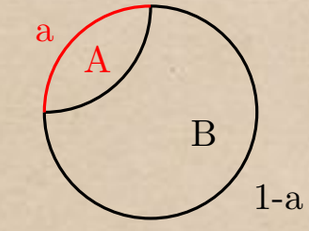
- Bekenstein-Hawking entropy (S_{BH}) $\propto$ horizon area.
- **Thermal AdS**: "ground state" **BTZ**: global "excitation"

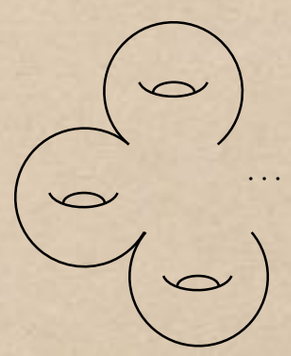


- Modular parameter $\tau = \theta + i\beta$ is ratio of 2 cycles, where β is inverse temperature, and $\theta = 0$ for non-rotating BTZ.

TEE Calculations

- Topological gravity $\rightarrow$ replica trick yields TEE $S_A = -\lim_{n\to 1}\frac{d}{dn}\mathrm{tr}\ \rho_A^n$
- **Thermal AdS**:

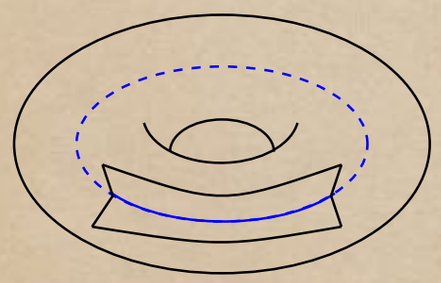


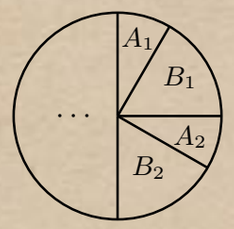


$$S_{TAdS}(a) \approx \left[96ke^{-2\pi\beta} + (96k-8)e^{-4\pi\beta} + O(e^{-6\pi\beta})\right](\pi a\cot(\pi a) - 1)$$

mutual information ~0 at low temperature

- **BTZ**: S_{BH} reproduced by cutting & gluing along horizon:





$$S_{BTZ}(A) = \frac{8\pi k}{\beta} - 2e^{-4\pi/\beta}\left(\frac{4\pi}{\beta} - 1\right) + O(e^{-6\pi/\beta})$$

mutual information ~S_{BH} at high temperature

TEE Calculations

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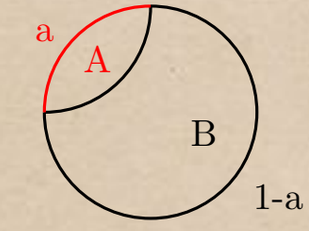
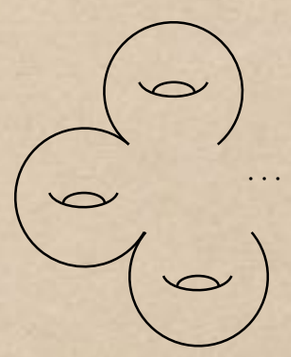
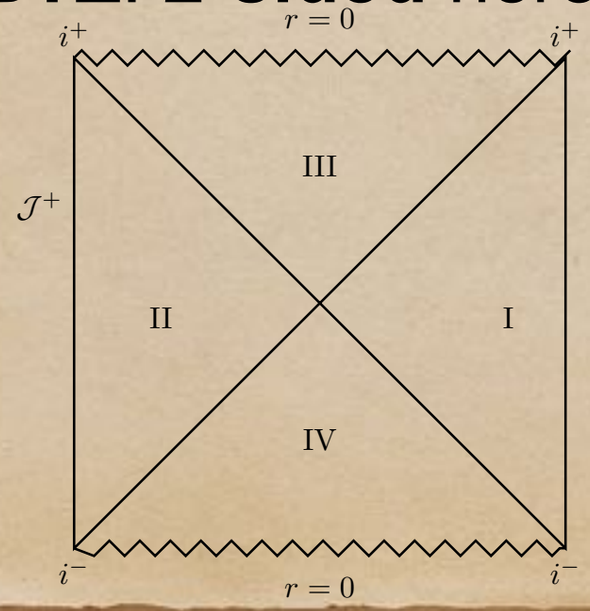




$$S_{TAdS}(a) \approx \left[96ke^{-2\pi\beta} + (96k-8)e^{-4\pi\beta} + O(e^{-6\pi\beta})\right](\pi a\cot(\pi a) - 1)$$

mutual information ~0 at low temperature

- **BTZ**: **2-sided** here, with Penrose diagram:



"Eternal black hole"

[J. M. Maldacena, JHEP 2003]

A 3d/bulk version of ER=EPR!

Modular Invariance

- Sum over geometries: $Z(\tau) = \sum_{c,d} Z_{c,d}(\tau) = \sum_{c,d} Z_{0,1}\left(\frac{a\tau + b}{c\tau + d}\right)$
- **Genus 1** modular-invariant partition function is a polynomial of j-invariant ($q = e^{2\pi i \tau}$):

$$J(\tau) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \dots$$

- At level $k = 1$, matching achieved by $Z(\tau) = J(\tau)$ for BTZ
- Conway-Norton conjecture on coefficients in j & dimensions of irreps of monster group $\mathbb{M}$:

$$196884 = 1 + 196883$$

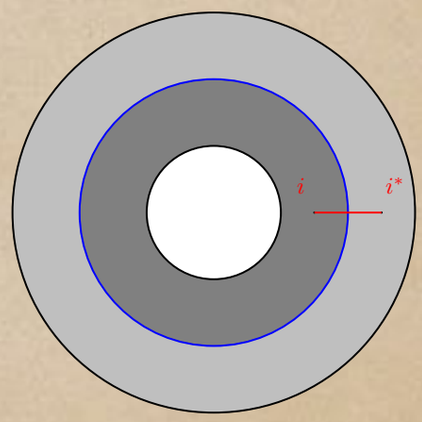
$$21493760 = 1 + 196883 + 21296876$$

- $\mathbb{M}$ has order ~8×10^{53}, but only 194 conjugacy classes/irreps

Moonshine Double States

- $J(q) = \sum_n \left(\sum_{i=0}^{193} m_i(n) d_i \right) q^n$, where $m_i(n) = \frac{d_i}{|\mathbb{M}|} \frac{e^{4\pi\sqrt{n}}}{\sqrt{2} n^{3/4}}$ is multiplicity of $\mathbb{M}$-modules.
- At high temperature, log of q^n coefficient agrees with S_{BH} at energy n [Witten, arXiv: 0706.3359]
- d_i appears twice: agree with wormhole described by “max. entangled” superpositions [J. Baez, Class. Quantum Grav. 2014]
- Quasiparticle pair labelled by irreps of $\mathbb{M}$:

$$|\Psi\rangle = \sum_{i=0}^{193} \frac{d_i}{\sqrt{\mathbb{M}}} |i, i^*\rangle$$



- The full system is $\rho = |\Psi\rangle\langle\Psi| \otimes \rho_{\mathrm{thermal}}$

Summary

- Thermal AdS TEE vanishes at low temperature;
- BTZ TEE = S_{BH}, bulk version of ER=EPR;
- After summing over geometries, **moonshine double state** is an analog of thermal field double (**TFD**) states of eternal AdS black holes, but in bulk, not on boundaries.

Thank you!