Higher permutohedra at one-loop

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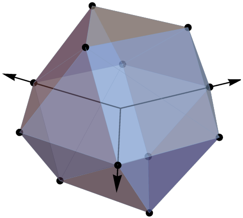
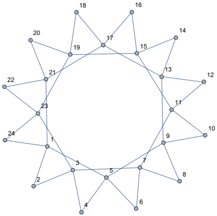
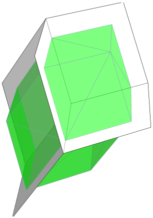






Outline

- **Objectives:** (1) sketch a new interpretation of Parke-Taylor factors using an algebra of permutohedral tilings $\mathcal{V}^n$, and (2) state basic but essential results about the interpretation.
- **Theorem[E]:** $\mathcal{V}^n$ is generated using *Minkowski sums of embedded trivalent graphs.* Any triangulation of an n-gon with a given cyclic vertex ordering defines a product of trivalent graphs; this product is independent of the triangulation!
- A paper describing the combinatorial framework is in preparation.
- Some back-history: Ph.D. thesis on symmetric group representations of generalized permutohedra; 1712.08520 (detailed study of characterisic functions of permutohedral cones, as *plates*), 1804.05460 (generalized permutohedra in the kinematic space).

From Parke-Taylor factors to permutohedral arrangements

- **Constructing the bridge:** define $x_{ij} = x_i - x_j$, set $\sigma = (1, 2, \ldots, n)$; put

$$\text{PT}(\sigma)(x) = \frac{1}{x_{12}x_{23}\cdots x_{n1}}.$$

- Setting $x_i = e^{-\varepsilon y_i}$, where ε is regarded as a formal dilation parameter, after the naive transformation $y_i \mapsto y_i - \frac{1}{n}\sum_{j=1}^{n} y_j$ we have...

$$\text{PT}(\sigma)(x) \mapsto \frac{x_1 \cdots x_n}{x_{12}x_{23}\cdots x_{n1}} = \frac{e^{-\varepsilon(y_1+\cdots+y_n)}}{e^{-\varepsilon y_{12}}e^{-\varepsilon y_{23}}\cdots e^{-\varepsilon y_{n1}}}.$$

- Taking the first two nonzero terms in the series expansion in ε we get:

$$\text{PT}(\sigma)(y)\varepsilon^{-n} + \frac{1}{12}\text{PT}_{\mathcal{L}}(\sigma)(y)\varepsilon^{-(n-2)} + \cdots,$$

where the coefficient of $\varepsilon^{-(n-2)}$ is the elementary symmetric function

$$\text{PT}_{\mathcal{L}}(\sigma)(y) = \sum_{1\le i<j\le n} \frac{1}{y_{12}\cdots\widehat{y_{i,i+1}}\cdots\widehat{y_{j,j+1}}\cdots y_{n1}}.$$

Algebras of Permutohedral Cones

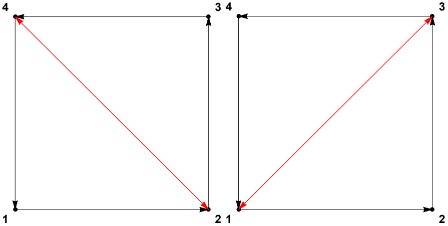
- We now interpret $\mathrm{PT}_{\mathcal{L}}(\sigma)$ in terms of an algebra $\mathcal{V}^n$ of permutohedra...
- Denote $u_{ij} = y_{ij}^{-1}$ and set(!) $v_{ijk} = u_{ij} + u_{jk} + u_{ki}$.
- These satisfy some relations:

$$\text{Antisymmetry: } u_{ij} = -u_{ji} \text{ and } v_{ijk} = v_{jki} = -v_{ikj}$$

$$\text{Linear straightening: } v_{ijk} - v_{jk\ell} + v_{k\ell i} - v_{\ell ij} = 0$$

$$\text{“Jacobi:” } u_{ij}u_{jk} + u_{jk}u_{ki} + u_{ki}u_{ij} = 0, v_{ijk}v_{ik\ell} + v_{ik\ell}v_{i\ell j} + v_{i\ell j}v_{ijk} = 0.$$

- **Proposition[E]: modulo the ideal generated by** u_{ij}^2 we have the square move $v_{124}v_{234} = v_{123}v_{134}$.



Triangulations

Theorem[E]. To each triangulation $\mathcal{I} = \{(i_1, j_2, k_1), \ldots, (i_{n-2}, j_{n-2}, k_{n-2})\}$, $(i_a < j_a < k_a)$ of an n-gon with cyclically ordered vertices $(1, \ldots, n)$, define

$$((\mathcal{I})) := v_{i_1 j_1 k_1} \cdots v_{i_{n-2} j_{n-2} k_{n-2}}.$$

Then, (1) **modulo the ideal generated by the u_{ij}^2's,** $((\mathcal{I}))$ is independent of the triangulation, and in fact $((\mathcal{I})) = \mathrm{PT}_{\mathcal{L}}(1, 2, \ldots, n)$. (2) There is a canonical basis for $\mathcal{V}^n$ with graded dimension the Stirling numbers of the first kind.

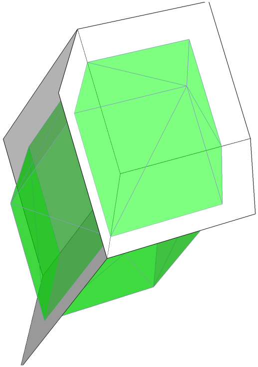
Example:

$$v_{123} v_{134} = (u_{12} + u_{23} + u_{31})(u_{13} + u_{34} + u_{41})$$

$$= u_{12} u_{23} + u_{12} u_{34} + u_{12} u_{41} + u_{23} u_{34} + u_{23} u_{41} + u_{34} u_{41}$$

Sketch of Proof of (1). Use $\exp(u_{ij}) = 1 + u_{ij}$ and $\exp(v_{ijk}) = 1 + v_{ijk}$ modulo u_{ij}^2's. Then, edges cancel additively!

Thank you!



Polytope for the Parke-Taylor factor $PT(1,2,3,4)$ (green), cut through by the six sheets for $PT_{\mathcal{L}}(1,2,3,4)$ (white).

References

- Early, N. "Generalized permutohedra in the kinematic space." 1804.05460.
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