Unifying Tree Super-Amplitudes in 6D: Branes, SYM, and SUGRA

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Main ideas

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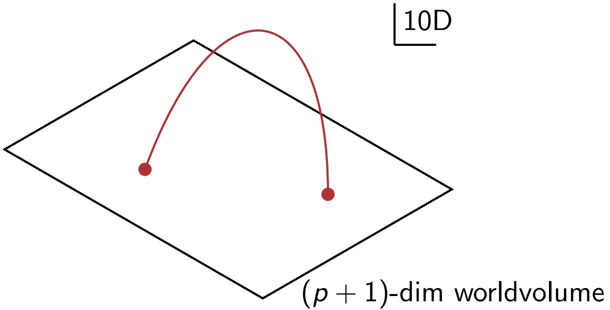
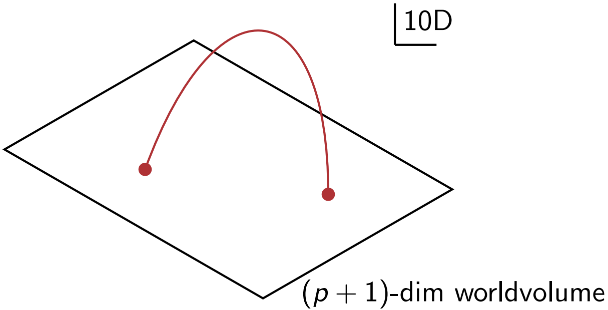
- Goal: Write down the complete tree-level S-matrix for maximally supersymmetric gauge theories, gravity, and effective field theory in six spacetime dimensions.
- The main tool is to boost Witten's twistor string from $4 \to 6$ dimensions. The n-particle amplitude is an integral over the n-punctured Riemann sphere (possibly with other moduli $\mathcal{M}$):

$$\mathcal{A}_n \sim \int \frac{d^n\sigma \, d\mathcal{M}}{\mathrm{Vol(G)}} \langle \text{String Correlation Function} \rangle$$

We find a unified description of 6D theories in this form.



- Quantization of open strings on Dirichlet p-brane $\rightarrow$ $p+1$ dimensional vector multiplet. Maximal SUSY for spins ≤ 1.



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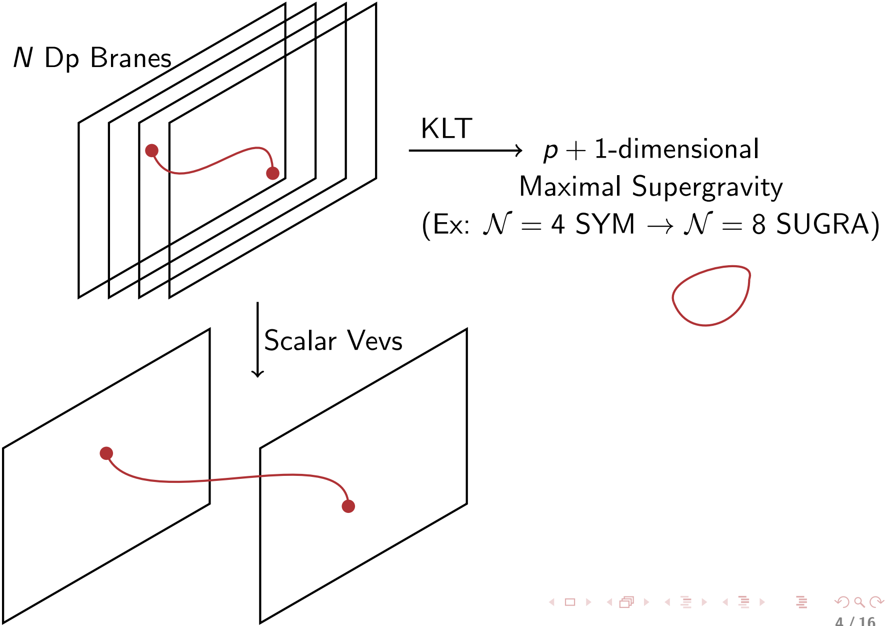
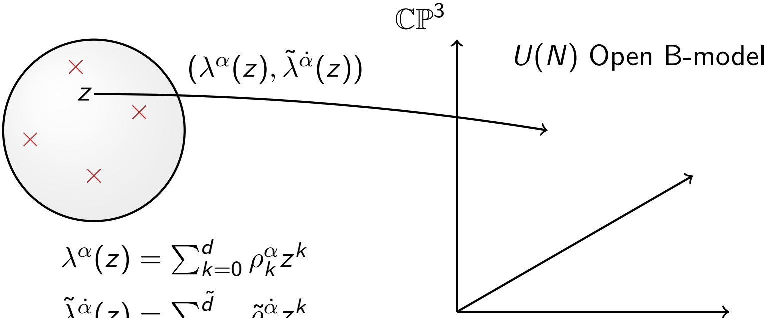
- For $\ell_s \rightarrow 0$, this is a free theory, but finite ℓ_s corrections give supersymmetric Dirac-Born-Infeld theory (brane action): $S \sim T \int d^{p+1}x \sqrt{-\det(g+F)}$

Multiple Branes → Super Yang-Mills



Twistor strings and rational maps

- Witten's observation: Scattering of $\mathcal{N} = 4$ SYM (field theory) computed exactly by a topological string theory! Open B-Model on supertwistor space.
- Amplitude supported on punctured D1 strings wrapping curves, integrate correlator over punctures and moduli of maps:



$$\lambda^\alpha(z) = \sum_{k=0}^{d} \rho_k^\alpha z^k$$

$$\tilde{\lambda}^{\dot{\alpha}}(z) = \sum_{k=0}^{\tilde{d}} \tilde{\rho}_k^{\dot{\alpha}} z^k$$

$$d + \tilde{d} = n - 2$$

$$d - \tilde{d} = \text{helicity violation}$$

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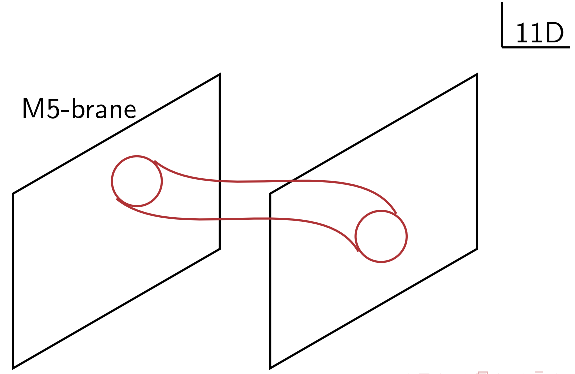
- Chiral and anti-chiral pseudo-real spinors: (p, q) SUSY. $(1, 1) \to$ D5-brane and 6D SYM. But $(2, 0) \to$ *self-dual (chiral)* 2-form gauge field $B_{\mu\nu}$. $H = dB = *H \to$ no candidate action!



The challenge:

- The Witten twistor string relied crucially on the 4D spinor helicity variables to construct maps. In 6D there are no helicity sectors due to the little group.
- The simplest 6D generalization turn out to be the single D5 and M5-brane EFTs. Maximal Yang-Mills and SUGRA are harder due to the structure of the maps.
- Apply to lower dimensions: 5D SYM/SUGRA and 4D Coulomb Branch amplitudes

Towards Rational Maps: Spinor Variables in 6D

- Momentum vectors can be described as bispinors of $Spin(5,1) \sim SU^*(4)$, $p^\mu \sim p^{AB}$ with $A, B = 1, \ldots, 4$.
- Little group $= SU(2) \times SU(2)$. We introduce λ^A_{ia} such that

$$p_i^{AB} = \langle \lambda_i^A \lambda_i^B \rangle = \epsilon_{ab} \lambda_i^{A,a} \lambda_i^{B,b} = \lambda_i^{A+} \lambda_i^{B-} - \lambda_i^{A-} \lambda_i^{B+}.$$

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- Lorentz invariants:

$$\langle \lambda_i^a \lambda_j^b \lambda_k^c \lambda_l^d \rangle = \epsilon_{ABCD} \lambda_i^{A,a} \lambda_j^{B,b} \lambda_k^{C,c} \lambda_l^{D,d},$$

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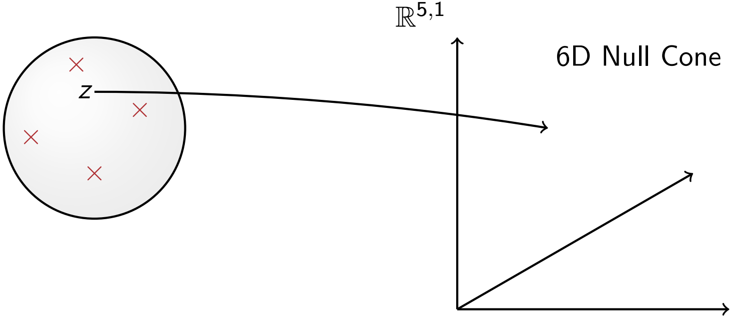
- E.g., 4 gluon scattering:

$$\mathcal{A}(g_1^{a\hat{a}}, g_2^{b\hat{b}}, g_3^{c\hat{c}}, g_4^{d\hat{d}}) = \frac{\langle 1^a 2^b 3^c 4^d \rangle [1^{\hat{a}} 2^{\hat{b}} 3^{\hat{c}} 4^{\hat{d}}]}{s_{12}\, s_{23}}.$$

$D = 6$ Rational Maps

Promote spinor variables to polynomials. Construct the null map:

$$z \in \mathbb{CP}^1 \longrightarrow p^{AB}(z) = \langle \rho^A(z), \rho^B(z) \rangle$$



The most natural choice consistent with $SL(2,\mathbb{C})$ is

$$\rho^{A,a}(z) = \sum_{k=0}^{d=\lceil \frac{n}{2} \rceil -1} \rho_k^{A,a} z^k.$$

where for odd n we require the degenerate condition $\rho_d^{A,a} = \omega^A \xi^a$!
These maps are to be determined by the condition

$$p_i^{AB} = \frac{\langle \rho^A(\sigma_i), \rho^B(\sigma_i) \rangle}{\prod_{j \neq i} \sigma_{ij}}$$

which also fixes the punctures $\{\sigma_i\}$ in $\mathbb{CP}^1$ (i.e. *Scattering Equations*)

6D Amplitude

We are now ready to construct the amplitudes for our favorite 6D theories by integrating over the moduli space of maps!

$$\mathcal{A}_{6D} = \int \frac{\prod d\sigma_i \, d\rho_k}{\mathrm{Vol}(G)(\prod \sigma_{ij})^2} \prod_{i=1}^{n} \delta^6 \left(p_i^{AB} - \frac{\langle \rho^A(\sigma_i), \rho^B(\sigma_i) \rangle}{\prod_{j \neq i} \sigma_{ij}} \right) \times \mathcal{I}_L \mathcal{I}_R$$

where $\mathrm{Vol}(G)$ stands for the redundancies of the moduli space. For odd n an enlarged symmetry group emerges. We find that the delta functions completely localize the integration variables on $(n-3)!$ points of the moduli.

M5, D5-Branes and SYM

The integrands $\mathcal{I}_{L,R}$ depend on the theory. They carry the fermionic components of the amplitude. This are defined in analogous way to the bosonic delta functions:

$$\Delta_F = \int \prod_{k=0}^{d} d\chi_k \prod_{i=1}^{n} \delta^4 \left(q_i^A - \frac{\langle \rho^A(\sigma_i), \chi(\sigma_i) \rangle}{\prod_{j \neq i} \sigma_{ij}} \right)$$

where q_i^A is now the supermomenta of the i-th particle.
We have half-integrands:

$$\mathcal{I}^{\mathcal{N}=(2,0)} = \mathrm{Pf}' A_n \times \Delta_F^2, \qquad \mathcal{I}^{\mathcal{N}=(1,1)} = \mathrm{Pf}' A_n \times \Delta_F \tilde{\Delta}_F$$

$$\mathcal{I}^{abelian} = (\mathrm{Pf}' A_n)^2, \qquad \mathcal{I}^{non-abelian} = \frac{\mathrm{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n})}{\sigma_{12}\sigma_{23} \cdots \sigma_{n1}}.$$

where $\mathrm{Pf}' A_n$ is constructed from minors of $(A_n)_{ij} := \frac{p_i \cdot p_j}{\sigma_{ij}}$

Use the half-integrands as building blocks for amplitudes:

$\int d\mu_{maps}\, \mathcal{I}_L \mathcal{I}_R$	$\mathcal{I}_L$	$\mathcal{I}_R$

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- $\mathcal{N} = (2,2)$ SUGRA is a double-copy of $\mathcal{N} = (1,1)$ SYM
- Perturbative amplitudes in non-abelian $\mathcal{N} = (2,0)$ theory should vanish; our formula computes some other non-abelian object with $\mathcal{N} = (2,0)$ on-shell supersymmetry

We find new amplitudes for mixed theories using the half-integrand:

$$\mathcal{I}^{semi-abelian} = \frac{\mathrm{Tr}(T^{a_1} T^{a_2} \cdots T^{a_k})}{\sigma_{12}\sigma_{23} \cdots \sigma_{k1}} (\mathrm{Pf} A_{k+1,\ldots,n})^2,$$

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$$\mathcal{A}^{D5-branes \oplus SYM} = \int d\mu_{maps}\, \mathcal{I}^{semi-abelian}\, \mathcal{I}^{\mathcal{N}=(1,1)}$$

gives an S-matrix of an interacting theory between the abelian and non-abelian sectors of D5-branes.

Even if you couldn't care less about 6D, we have something for you:
Embed 4D massive momenta into 6D massless ones as follows

$$\lambda^{A,a} = \begin{pmatrix} \frac{m\mu_\alpha}{\langle\mu\lambda\rangle} & \lambda_\alpha \\ \tilde{\lambda}^{\dot{\alpha}} & \frac{m\tilde{\mu}^{\dot{\alpha}}}{[\tilde{\lambda}\tilde{\mu}]} \end{pmatrix} \quad \Longrightarrow \quad p_{\alpha\dot{\alpha}} = \lambda_\alpha\tilde{\lambda}_{\dot{\alpha}} + m^2\frac{\mu_\alpha\tilde{\mu}_{\dot{\alpha}}}{\langle\lambda\mu\rangle[\tilde{\lambda}\tilde{\mu}]}.$$

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- We can do concrete calculations, for instance, 4-pt amplitude of W-bosons and gluons:

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- Leaves many applications for computing loop integrands

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  MTH, J.H. Schwarz, C. Wen [hep-th/1710.02170]
- *"The S Matrix of 6D Super Yang–Mills and Maximal Supergravity from Rational Maps"*
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