Lectures on Superstring Amplitudes

Part 1: Bosonic String

Eric D'Hoker
Mani L. Bhaumik Institute for Theoretical Physics
University of California, Los Angeles

Center for Quantum Mathematics and Physics - 2018
Amplitudes 2018 Summer School

Outline of lectures

- **Lecture 1**

  Bosonic strings and conformal field theory

- **Lecture 2**

  Superstring amplitudes

- **Lecture 3**

  Low energy effective interactions, modular graph functions

Strings

- **A string is a 1-dimensional object**
  - open string = topology of an interval;
  - closed string = topology of a circle;
  - physical size Planck length $\ell_P \approx 10^{-33}\mathrm{cm} \approx 10^{-19}\times$ size of the proton.

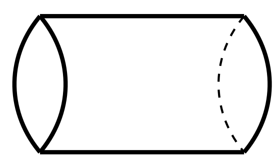
- **Ultimate goal: unified theory of particle physics and gravity**
  - elementary particles correspond to strings and their excited states;
  - consistently with quantum mechanics and general relativity;
  - remarkably unique structure.

- **Immediate goal: relating string amplitudes and field theory amplitudes**
  - at distance scales larger than the Planck length (low energy) a string effectively behaves as a point particle
  - string theory exhibits powerful structure of amplitudes

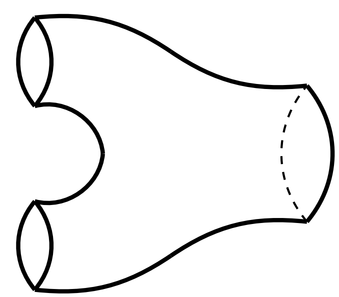
String Topology

- **Consistent interacting string theories**
  - only closed strings (Type IIA,B and heterotic)
  - closed and open strings (Type I)
  - Type II theories have open strings in the presence of D-branes
- **Strings live in a physical space-time M**
  - M may be a manifold or an orbifold (with mild isolated singularities)
  - superstring theory predicts 10-dim
  - but space-time visible to us is 4-dim. $\Rightarrow$ requires "compactification"
- **Under time-evolution strings sweep out a 2-dim. surface**

closed strings



time-evolution
(freely propagating)



basic interaction
(purely topological)

Perturbative String Amplitudes

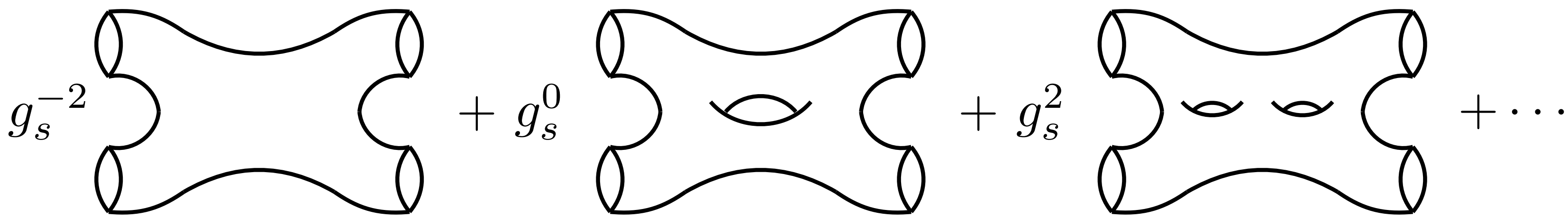
- **Quantum probability "scattering" amplitudes**

  = Feynman functional integral/sum over all surfaces
  with given boundary components for initial and final strings

- **Closed oriented string perturbation theory**

  – The only remaining topological characterization is the genus $h \geq 0$

  - probability amplitude includes sum over all genera
  - weighed by a factor g_s^{2h-2} where g_s is the "string coupling"



  – genus h = number of "loops"

Structure of string amplitudes

- **Perturbative part of string amplitude decomposes into a sum over topologies**

$$\mathcal{A}_{\text{perturbative}} = \sum_{h=0}^{\infty} g_s^{2h-2} \times \mathcal{A}^{(h)}$$

  – $\mathcal{A}^{(h)}$ is the amplitude at genus h

  – The perturbative expansion in g_s is asymptotic but not convergent (just as in field theory)

- **Non-perturbative part** (not considered here)

  ⋆ instantons $\approx e^{-1/g_s^2}$
  ⋆ D-branes contribute $\approx e^{-1/g_s}$.

String Data (closed oriented bosonic strings)

- **Assume fixed space-time M, with fixed metric G**
  - Physical space-time has Minkowski signature metric G
  - Starting point for string theory is often a Riemannian metric (if needed to be analytically continued to Minkowski signature)
- **The 2-dimensional worldsheet Σ is mapped into space-time M**
  - The space of all such maps $x : \Sigma \to M$ is denoted $\mathrm{Map}(\Sigma)$.
- **Riemannian metric G induces a Riemannian metric $x^*(G)$ on Σ**
  - Hence Σ is a Riemann surface (i.e. complex manifold with holó transition functions)
- **Polyakov formulation invokes an independent metric**
  - Riemannian metric g on Σ
  - Denote the ∞-dim. Riemannian manifold of such metrics by $\mathrm{Met}(\Sigma)$
  - String amplitude at fixed genus h obtained by weighed sum over g, x

$$\mathcal{A}^{(h)} = \int_{\mathrm{Met}(\Sigma)} Dg \int_{\mathrm{Map}(\Sigma)} Dx \; e^{-I_G[x,g]}$$

The worldsheet action I_G and the measure Dx

- **Basic Criteria**
  - Intrinsic = invariant under "reparametrizations" Diff(Σ) of Σ
  - lead to a well-defined QFT (renormalizable)

- **e.g. Non-linear sigma model action** with Riemannian metric G

$$I_G[x,g] = \frac{1}{\alpha'} \int_\Sigma d^2\xi \sqrt{g}\, g^{mn} \partial_m x^\mu \partial_n x^\nu G_{\mu\nu}(x)$$

$m,n = 1,2$ worldsheet indices

$\mu,\nu = 1,\cdots,D$ space-time Einstein indices

- **The measure is governed by the L^2-norm**

$$\|\delta x\|^2_G = \int_\Sigma d^2\xi \sqrt{g}\, \delta x^\mu\, \delta x^\nu\, G_{\mu\nu}(x)$$

  - manifestly intrinsic
  - renormalizable in a generalized sense (the metric G is renormalized)

Weyl(Σ)-invariance

- **Weyl transformations:** $g_{mn} \to e^{2\sigma} g_{mn}$ **leaving** x^μ **and** G **unchanged**

- **The classical action** I_G **is Weyl-invariant for any metric** G
  - but the measure Dx is not Weyl-invariant
  - which gives rise to a "Weyl-anomaly"
    - $=$ symmetry of classical action not preserved by quantization

- **The action** I_G **defines a conformal quantum field theory**

$$e^{-W_G[g]} = \int_{\mathrm{Map}(\Sigma)} Dx \; e^{-I_G[x,g]}$$

  - provided W_G is Diff(Σ)-invariant
  - obeys the following Ward identity under Weyl transformations

$$\delta W_G[g] = \frac{c}{24} \int_\Sigma d^2\xi \sqrt{g}\, R_g \, \delta\sigma$$

  - where R_g is the scalar curvature of the metric g on the surface Σ

- **The measure** Dg **is not Weyl-invariant, but the combined amplitude**
  - is Weyl invariant for central charge $c = 26 = \dim(M)$
  - later we shall see for the superstring $D = 10 = \dim(M)$

Conformal Field Theory

- **Stress tensor encodes response of field theory to change in metric**

$$T^c_{mn} = \frac{\delta W_G[g]}{\sqrt{g}\,\delta g^{mn}} \qquad\qquad T^c_{mn} = T^c_{nm}$$

  - Diff(Σ)-invariance requires "a conserved stress tensor" $\nabla^m T^c_{mn} = 0$
  - Weyl anomaly requires $g^{mn} T^c_{mn} = -\frac{c}{12} R_g$

- **Traceless stress tensor** T_{mn} **obtained by adding a local counter-term**
  - In local complex coordinates $(z, \tilde{z})$ we have $T_{z\tilde{z}} = T_{\tilde{z}z} = 0$ and

$$T_{zz} = T^c_{zz} + \frac{c}{6}\left(2\partial_z \Gamma^z_{zz} - (\Gamma^z_{zz})^2\right) \qquad\qquad \Gamma^z_{zz} = \partial_z \ln g_{z\tilde{z}}$$

  - Successive derivatives of W in g_{mn} give correlators of T_{mn}
  - Their singular part is governed by the OPE and the Ward identities

$$T_{zz} T_{ww} = \frac{c/2}{(z-w)^4} + \frac{2T_{ww}}{(z-w)^2} + \frac{\partial_w T_{ww}}{z-w} + \text{regular}$$

  - The mode expansion $T_{zz} = \sum_m z^{-2-m} L_m$ gives the Virasoro algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$

Negative norm states

- **Consider flat Minkowski $M = \mathbb{R}^{26}$ with metric $\eta = \mathrm{diag}(- + \cdots +)$**
  - Maps $x : \Sigma \to M$ satisfy Laplace equation $\partial_{\bar z} \partial_z x^\mu = 0$ for $\mu = 1, \cdots, 26$
  - Concentrate on holomorphic field

$$\partial_z x^\mu = \sum_{m \in \mathbb{Z}} x_m^\mu \, z^{-m-1} \qquad [x_m^\mu, x_n^\nu] = m \, \delta_{m+n,0} \, \eta^{\mu\nu} \qquad (x_n^\mu)^\dagger = x_{-n}^\mu$$

  - Similarly anti-holomorphic field $\partial_{\bar z} x^\mu$ produces modes $\tilde{x}^\mu$

- **Single string ground state $|0,k\rangle$ labeled by its momentum k satisfies**

$$x_0^\mu |0,k\rangle = k^\mu |0,k\rangle \qquad x_m^\mu |0,k\rangle = 0 \text{ for } m > 0$$

  - Fock space (holo sector) generated by linear combinations of

$$x_{m_1}^{\mu_1} \cdots x_{m_p}^{\mu_p} |0,k\rangle \qquad m_1, \cdots, m_p < 0$$

  - Lowest excited state $\varepsilon_\mu(k) x_{-1}^\mu |0,k\rangle$ has norm

$$\| \varepsilon_\mu(k) x_{-1}^\mu |0,k\rangle \|^2 = \varepsilon_\mu(k) \varepsilon_\nu(k) \eta^{\mu\nu} \, \| |0,k\rangle \|^2$$

  - component $\varepsilon^\mu = \delta^{\mu,0}$ produces *negative norm state* (assuming $\| |0,k\rangle \|^2 > 0$)
    $=$ inconsistent with quantum mechanical probability interpretation

Eliminating negative norm states – conformal symmetry

- **Conformal symmetry guarantees the existence of Virasoro algebra**

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$

  – for the bosonic string $c = 26$ and

$$L_m = \sum_{n\in\mathbb{Z}} \frac{1}{2} x_{m-n} \cdot x_n \qquad\qquad L_0 = \frac{1}{2}x_0^2 + \sum_{n\in\mathbb{N}} x_{-n} \cdot x_n$$

- **A state $|\psi\rangle$ is "physical" if $(L_0 - 1)|\psi\rangle = L_m|\psi\rangle = 0$ for $m \in \mathbb{N}$**
  – Eliminates all negative norm states;
  – Decouples all null states produced by gauge transformations;

  – e.g. on states $|\psi\rangle = \varepsilon(k) \cdot x_{-1}|0,k\rangle$
    ⋆ L_1 constraint imposes $k \cdot \varepsilon(k) = 0$
    ⋆ L_0 constraint imposes $k^2 = 0$
    ⋆ L_m constraints are automatic for $m \geq 2$ for this particular state
  – the state $|0,k\rangle$ itself is a tachyon (to be absent in the superstring)

$\Rightarrow$ **Negative norm and null states eliminated by conformal symmetry**

Conformal symmetry in curved space-times

- **Condition for Weyl-invariance on the metric G**
  - Infinitesimal Weyl variation for arbitrary G to one-loop order in α'

$$\delta W_G[g] = \int_\Sigma d^2\xi \sqrt{g}\, g^{mn} \partial_m x^\mu \partial_n x^\nu R_{\mu\nu}(x) \delta\sigma + \cdots + \mathcal{O}(\alpha')$$

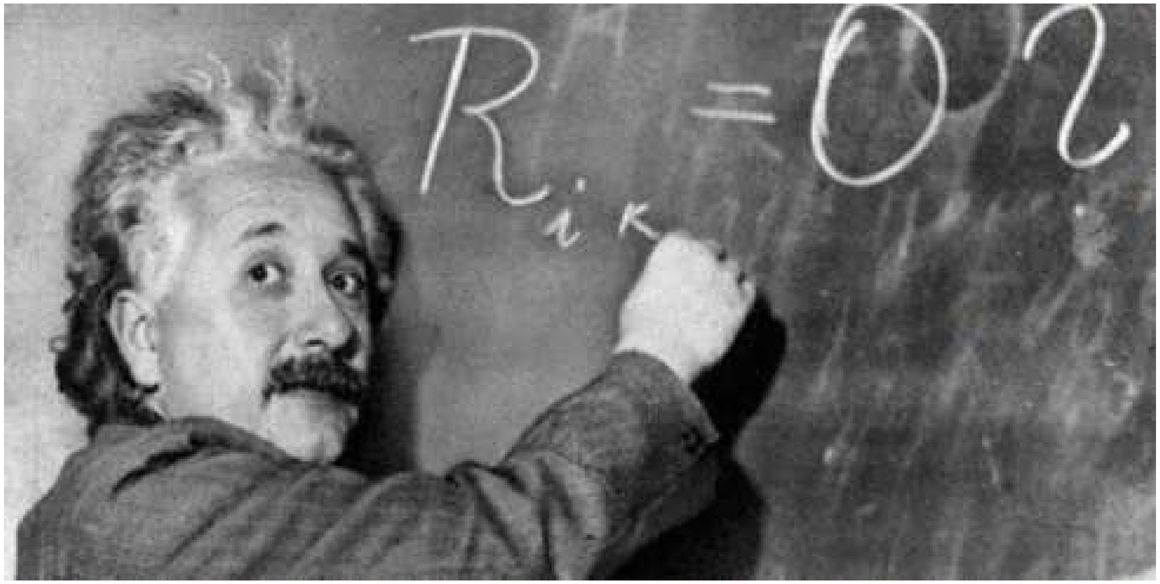
    where $R_{\mu\nu}$ is the Ricci tensor of the metric $G_{\mu\nu}$
  - Thus, to leading order in α' conformal invariance requires $R_{\mu\nu} = 0$

Conformal symmetry in curved space-times

- **Condition for Weyl-invariance on the metric G**
  - Infinitesimal Weyl variation for arbitrary G to one-loop order in α'

$$\delta W_G[g] = \int_\Sigma d^2\xi \sqrt{g}\, g^{mn} \partial_m x^\mu \partial_n x^\nu R_{\mu\nu}(x) \delta\sigma + \cdots + \mathcal{O}(\alpha')$$

    where $R_{\mu\nu}$ is the Ricci tensor of the metric $G_{\mu\nu}$
  - Thus, to leading order in α' conformal invariance requires $R_{\mu\nu} = 0$



Vertex operators

- **Small fluctuations in the metric are gravitons**
  - A string couples to N gravitons in flat space by slightly perturbing the metric

$$G_{\mu\nu}(x) = \eta_{\mu\nu} + \sum_{i=1}^{N} \varepsilon_{i\mu\nu}(k_i)\, e^{ik_{i\mu}x^\mu} + \mathcal{O}(\varepsilon^2)$$

  - conformal invariance requires G to satisfy the linearized Einstein equations

$$k_i^2 = 0 \qquad\qquad k_i^\mu \varepsilon_{i\mu\nu}(k_i) = 0 \qquad\qquad \text{for } i = 1, \cdots, n$$

- **Vertex operator formulation is obtained by expanding in powers of** ε_i

$$\mathcal{A} = \sum_{h=0}^{\infty} g_s^{2h-2} \int_{\mathrm{Met}(\Sigma)} Dg \int_{\mathrm{Map}(\Sigma)} Dx\ \mathcal{V}_1[x,g] \cdots \mathcal{V}_N[x,g]\, e^{-I_\eta[x,g]}$$

  - where the vertex operator for an on-shell physical graviton is given by

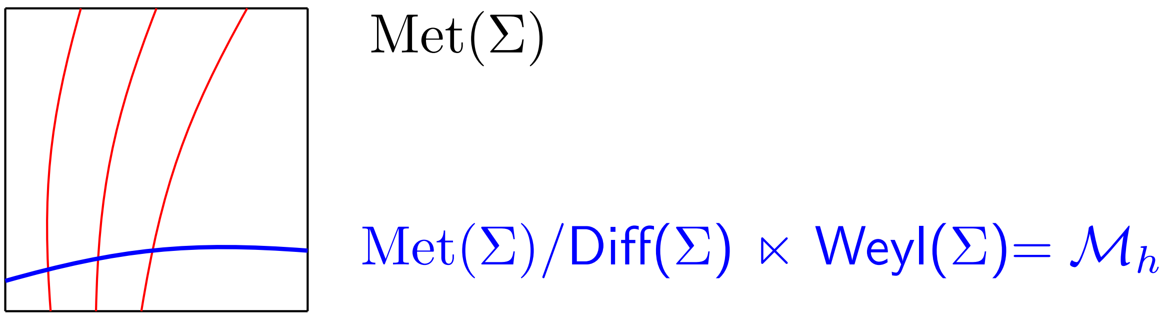
$$\mathcal{V}_i[x,g] = \varepsilon_{i\mu\nu}(k_i) \int_\Sigma d^2\xi \sqrt{g}\, g^{mn} \partial_m x^\mu \partial_n x^\nu\, e^{ik_\mu x^\mu}$$

  - On-shell conditions $k_i^2 = k_i \cdot \varepsilon_i = 0$ guarantee conformal invariance

Diff(Σ) $\ltimes$ Weyl(Σ) and Moduli space

- **Fix topology of Σ**
  - Diff(Σ) re-parametrizes ξ^m on Σ by vector field $\delta\xi^m = -\delta v^m$
    $$\delta g_{mn} = \nabla_m \delta v_n + \nabla_n \delta v_m$$
  - Weyl (Σ) $\quad \delta g_{mn} = 2\delta\sigma\, g_{mn}$ with $\delta\sigma$ an arbitrary real function of Σ

- **Orbits of Diff(Σ) $\ltimes$ Weyl(Σ) acting on the space Met(Σ)**



- **Moduli space $\mathcal{M}_h$ of compact Riemann surfaces of genus h** (no boundaries)
  = space of conformal structures (= space of complex structures)

$$\dim_{\mathbb{C}} \mathcal{M}_h = \begin{cases} 0 & h = 0 \\ 1 & h = 1 \\ 3h-3 & h \geq 2 \end{cases}$$

Some trivial moduli spaces

- **Given an infinitesimal δg_{mn} can one solve for $\delta\sigma$ and δv_m ?**

$$\delta g_{mn} = 2\delta\sigma g_{mn} + \nabla_m \delta v_n + \nabla_n \delta v_m$$

  - Eliminate the trace part by choosing $\delta\sigma = g^{mn}\delta g_{mn} + \nabla_m \delta v^m$
  - In local complex coordinates $(z, \tilde{z})$, remaining eqs for traceless part

$$\delta g_{zz} = \nabla_z v_z \qquad\qquad \delta g_{\tilde{z}\tilde{z}} = \nabla_{\tilde{z}} v_{\tilde{z}}$$

  - Integrability automatic since ∇_z and $\nabla_{\tilde{z}}$ act on different functions

  $\Rightarrow$ locally, or in any simply connected set, you can always solve

- **The sphere S^2 has no moduli (compact)**
  - Its stereographic projection onto $\mathbb{C}$ admits a globally conformally flat metric

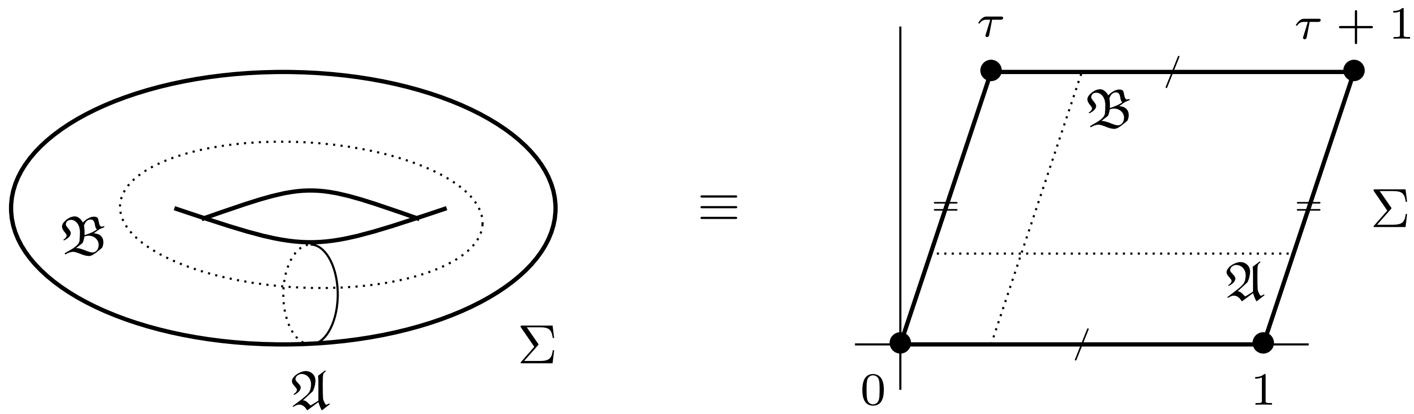
$$ds^2 = \frac{|dz|^2}{(1+|z|^2)^2}$$

- **The Poincaré upper half plane $\mathcal{H}$ has no moduli (non-compact)**

$$ds^2 = \frac{|dz|^2}{(\mathrm{Im}\, z)^2} \qquad\qquad \mathrm{Im}\, z > 0$$

Moduli deformations of the torus

- **The torus may be viewed as the product of two circles $\mathfrak{A}$ and $\mathfrak{B}$**
  - The ratio of their lengths and relative angle provide two real moduli
  - equivalently represented by parallelogram in $\mathbb{C}$ with sides pairwise identified



  - The complex number τ contains the information of relative lengths and angle

- **Constant metric deformations equivalently provide a complex modulus**
  - translation invariance on the circles induces translation invariance on the torus
  - by translation invariance, metric is constant on Σ
  - constant trace-part of δg_{mn} eliminated by constant σ
  - but constant $\delta g_{zz} = \partial_z v_z$ has no *periodic* solutions v_z

    $\Rightarrow$ constant δg_{zz} provides the deformation of the complex modulus of the torus.

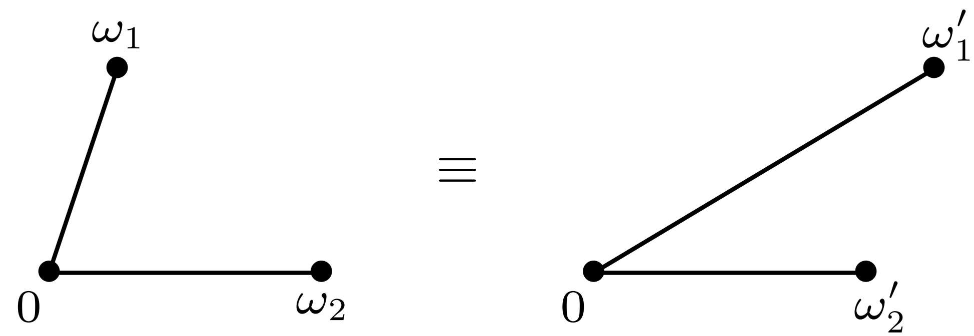
Moduli space of the torus

- **Oriented Riemann surfaces: cycles $\mathfrak{A}$ and $\mathfrak{B}$ ordered**
  - equivalently choose orientation $\tau \in \mathcal{H}_1 = \{\tau \in \mathbb{C}, \mathrm{Im}(\tau) > 0\}$

- **Space of inequivalent tori = space of inequivalent lattices** $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$
  - but different values of τ may give the same lattice

$$\omega_1' = a\,\omega_1 + b\,\omega_2$$
$$\omega_2' = c\,\omega_1 + d\,\omega_2$$

$$\tau = \omega_1/\omega_2$$
$$\tau' = (a\tau + b)/(c\tau + d)$$



  - identical lattices requires $\Lambda_{\tau'} \subset \Lambda_\tau$ and $\Lambda_\tau \subset \Lambda_{\tau'}$
  - so that $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$
  - generated by $\tau \to \tau + 1$ and $\tau \to -\tau^{-1}$

- **Moduli space of tori = space of inequivalent lattices =** $\mathcal{H}_1/SL(2,\mathbb{Z})$
  - standard fundamental domain

$$\mathcal{H}_1/SL(2,\mathbb{Z}) \equiv \left\{\tau \in \mathcal{H}_1,\, |\tau| \geq 1,\, |\mathrm{Re}(\tau)| \leq \frac{1}{2}\right\}$$

Decomposing the measure Dg

- **At any point $g \in \mathrm{Met}(\Sigma)$ the measure Dg factors**

$$
\begin{array}{ccccccccc}
Dg & = & Z_g & \times & D\sigma & \times & Dv & \times & d\mu_{\mathcal{M}_h} \\
 & & \text{Jacobian} & & \text{Weyl} & & \text{Diff}_0 & & \mathcal{M}_h
\end{array}
$$

  - infinitesimal Weyl $\delta g_{mn} = \delta\sigma\, g_{mn}$
  - infinitesimal Diff_0 $\delta g_{mn} = \nabla_m \delta v_n + \nabla_n \delta v_m$
  - infinitesimal moduli deformations δg_{mn}

- **Goal**
  - compute Z_g
  - formulate Z_g in terms of ghosts
  - omit volume factors $D\sigma\, Dv$ of the group $\text{Diff}^+(\Sigma) \ltimes \text{Weyl}(\Sigma)$

- **To decompose Dg we study tensor spaces (alias line bundles) on Σ**

Tensor Spaces - Line Bundles on Σ

- **A one-form $\phi = \phi_z dz + \phi_{\tilde z} d\tilde z$ on Σ decomposes into $K \oplus \bar K$**

  $K = \{\phi_z dz\}$ is the (space of sections of the) canonical bundle on Σ

  for $m \in \mathbb{Z}$ define $K^m = \{\phi_{z \ldots z} dz^m\}$ and $\bar K^m = \{\phi_{\tilde z \ldots \tilde z} d\tilde z^m\} \approx K^{-m}$

- **L^2 inner product for $\phi_1, \phi_2 \in K^m$**

$$(\phi_1, \phi_2) = \int_\Sigma d\tilde z dz \sqrt{g}\, (g_{z\tilde z})^{-m} \phi_1^* \phi_2$$

  The spaces K^m and K^n with $m \neq n$ are mutually orthogonal

- **Covariant derivative on $\phi \in K^m$ decomposes $\nabla \phi = \nabla_z^{(m)} \phi + \nabla_{\tilde z}^{(m)} \phi$**

  $\nabla_z^{(m)} : K^m \to K^{m+1}$ mutual adjoint operators $(\nabla_z^{(m)})^\dagger = -\nabla^z_{(m+1)}$

  $\nabla^z_{(m)} : K^m \to K^{m-1}$ with $\nabla_{\tilde z}^{(m)} = g_{z\tilde z} \nabla^z_{(m)}$

- **Riemann-Roch and Vanishing Theorems**

$$\dim_{\mathbb{C}} \mathrm{Ker}\, \nabla_{\tilde z}^{(m)} - \dim_{\mathbb{C}} \mathrm{Ker}\, \nabla_{\tilde z}^{(1-m)} = (2m-1)(h-1)$$

  $\mathrm{Ker} \nabla_{\tilde z}^{(m)} = 0$ for $h \geq 2$ and $m \leq -1$ (no holó vector fields for $h \geq 2$)

  $\mathrm{Ker} \nabla_{\tilde z}^{(m)} = 0$ for $h = 0$ and $m \geq 1$ (no holó forms on the sphere)

Decomposing the tangent space to Met(Σ)

- **Orthogonal decomposition of** $T_g(\mathrm{Met}(\Sigma))$

$$T_g(\mathrm{Met}(\Sigma)) = \{\delta\sigma\, g_{z\bar z}\} \oplus \left\{\delta g_{zz} = g_{z\bar z}\,\delta\eta_z{}^{\bar z}\right\} \oplus \{\delta g_{\bar z\bar z} = g_{z\bar z}\,\delta\eta_{\bar z}{}^{z}\}$$

$$\delta\sigma \in K^0 \qquad \delta\eta_z{}^{\bar z} \in K \otimes \bar K^{-1} \qquad \delta\eta_{\bar z}{}^{z} \in \bar K \otimes K^{-1}$$

- **Diff$_0$ acts by** $\delta\eta_z{}^{\bar z} = \nabla_z^{(1)}\delta v^{\bar z}$
  - For $h \geq 1$, the range of the operator $\nabla_z^{(1)}$ is NOT all of $K \otimes \bar K^{-1}$
  - The orthogonal complement of the range of $\nabla_z^{(1)}$ is given by

$$\mathrm{Range}\,\nabla_z^{(1)} \oplus \mathrm{Ker}\big(\nabla_z^{(1)}\big)^\dagger = K \otimes \bar K^{-1} \approx K^2$$

- **Holomorphic quadratic differentials** $\phi^j \in \mathrm{Ker}\nabla_{\bar z}^{(2)} \approx \mathrm{Ker}\big(\nabla_z^{(1)}\big)^\dagger$
  - Hence we may identify $\mathrm{Ker}\nabla_{\bar z}^{(2)} = T^*_{(1,0)}(\mathcal{M}_h)$
  - One-forms $\delta m^j \in T^*_{(1,0)}(\mathcal{M}_h)$ given by linear forms on $\bar K \otimes K^{-1}$

$$\delta m^j = (\delta\eta, \phi^j) = \int_\Sigma d\bar z dz\,\delta\eta_{\bar z}{}^{z}\,\phi^j_{zz}$$

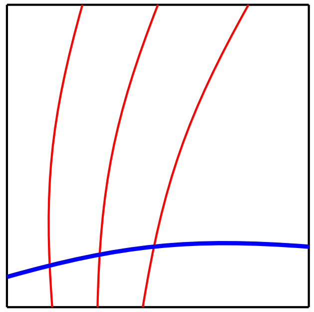
  - Weyl-invariant pairing and vanishes on $\delta\eta \in \mathrm{Range}\,\nabla_z^{(1)}$
  - Riemann-Roch and Vanishing give $\dim_{\mathbb C}\mathcal{M}_h = 3h-3$ for $h \geq 2$

Decomposing the measure Dg (cont'd)

- **Parametrize $\mathcal{M}_h$ by a slice in $\mathrm{Met}(\Sigma)$ transverse to Weyl $\ltimes$ Diff$_0$**

orbits of Diff(Σ) $\ltimes$ Weyl(Σ)

$m^j, \tilde m^j$ local coordinates on $\mathcal{M}_h$



- **Carry out a change of integration variables**

$$T_g(\mathrm{Met}(\Sigma)) = \{\delta\sigma\, g_{z\bar z}\} \oplus \{\delta\eta_z{}^{\bar z}\} \oplus \{\delta\eta_{\bar z}{}^z\}$$

 - Orthogonality implies that the measure factorizes $Dg = D\sigma\, D\eta\, D\bar\eta$
 - The change of variables is given by (repeated indices j are summed)

$$\delta\eta_{\bar z}{}^z = \nabla_{\bar z}^{(-1)} \delta v^z + (\mu_j)_z{}^{\bar z} \delta m^j \qquad (\mu_j)_{\bar z}{}^z = g^{z\bar z} \frac{\partial g_{\bar z\bar z}}{\partial m^j}$$

$$\delta\eta_z{}^{\bar z} = \nabla_z^{(1)} \delta v^{\bar z} + (\tilde\mu_j)_z{}^{\bar z} \delta\tilde m^j \qquad (\tilde\mu_j)_z{}^{\bar z} = g^{z\bar z} \frac{\partial g_{zz}}{\partial \tilde m^j}$$

Ghosts

- **Use standard rules to introduce ghosts for the determinant**
  - gauge transformations $(\delta v^z, \delta v^{\tilde z}) \to (c^z, \tilde c^{\tilde z})$ Grassmann-odd ghosts
  - conjugate $(\delta \eta_z{}^{\tilde z}, \delta \eta_{\tilde z}{}^z) \to (b_{zz}, \tilde b_{\tilde z \tilde z})$ Grassmann-odd anti-ghosts
  - extended ghost action

$$\int_\Sigma d^2 z \Big[b_{zz} \left(\partial_{\tilde z} c^z + \mu_j \, \delta m^j \right) + \tilde b_{\tilde z \tilde z} \left(\partial_z \tilde c^{\tilde z} + \tilde \mu_j \, \delta \tilde m^j \right) \Big]$$

  - Here $\delta m^j, \delta \tilde m^j$ are differential one-forms which are Grassmann odd

- **Integrating out $\delta m^j, \delta \tilde m^j$ gives the standard ghost representation**

$$\int D(x^\mu, b, \tilde b, c, \tilde c) \mathcal{V}_1 \cdots \mathcal{V}_N e^{-I_G - I_{gh}} \prod_j \delta(\langle b, \mu_j \rangle) \delta(\langle \tilde b, \tilde \mu_j \rangle) dm^j d\tilde m^j$$

  - where I_{gh} is the standard ghost action

$$I_{gh} = \int_\Sigma d^2 z \Big[b_{zz} \partial_{\tilde z} c^z + \tilde b_{\tilde z \tilde z} \partial_z \tilde c^{\tilde z} \Big]$$

  - gauge fixed formulation has BRST invariance
  - for the sphere and the torus, quotient out by conformal automorphisms

Bosonic string has tachyon and no fermions: unphysical

- **Warm-up : tree-level tachyon scattering amplitude**
  - tachyon vertex operator $\mathcal{V}(k_i) = \int_\Sigma d^2 z_i \sqrt{g(z_i)} :e^{ik_i \cdot x(z_i)}:$
  - scalar Green function on the sphere with metric $|dz|^2/(1+|z|^2)^2$

$$\langle x^\mu(z) x^\nu(w)\rangle = \eta^{\mu\nu} G(z,w) \qquad\qquad G(z,w) = -\ln \frac{|z-w|^2}{(1+|z|^2)(1+|w|^2)}$$

- **Sphere has no moduli, ghost and scalar partition functions are constant**

$$\langle \prod_{i=1}^N d^2 z_i \sqrt{g(z_i)} :e^{ik_i\cdot x(z_i)}: \rangle = \prod_{i=1}^N d^2 z_i \prod_{i<j}^N |z_i - z_j|^{\alpha' k_i \cdot k_j}$$

  - Integrand invariant under $z_i \to (\alpha z_i + \beta)/(\gamma z_i + \delta)$ with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{C})$
  - Factor out volume of $SL(2,\mathbb{C})$ by fixing $z_N = \infty, z_{N-1} = 1, z_{N-2} = 0$

- **The 4-tachyon amplitude with** $s_{ij} = -\alpha'(k_i + k_j)^2/4$

$$\frac{1}{g_s^2} \int_\Sigma d^2 z \, |z|^{\alpha' k_1 \cdot k_2} |z-1|^{\alpha' k_1 \cdot k_3} = \frac{\Gamma(-1-s)\Gamma(-1-t)\Gamma(-1-u)}{g_s^2 \, \Gamma(2+s)\Gamma(2+t)\Gamma(2+u)}$$

  - Tachyon poles at $s,t,u = -1$

Kawai-Lewellen-Tye (KLT) relations

- **Tree-level closed string amplitudes are bilinears in open string amplitudes**
  - Closed string amplitudes on the sphere, vertex operators in interior
  - Open string amplitude on upper half plane, vertex operators on boundary
  - Consider open and closed string 4-tachyon amplitudes

$$\mathcal{A}^{(0)}_{\rm open}(s,t) = \int_0^1 d\xi |\xi|^{k_1 \cdot k_2} |1-\xi|^{k_2 \cdot k_3} \qquad \mathcal{A}^{(0)}_{\rm closed}(s,t,u) = \int_{S^2} d^2 z |z|^{2k_1 \cdot k_2} |1-z|^{2k_2 \cdot k_3}$$

  - Parametrize $z = \alpha + i\beta$ then z-integrand is analytic function of β with branch points at $\beta = \pm i\alpha$ and $\beta = \pm i(1-\alpha)$
  - Deform β-contour from real to imaginary axis, but pick up phases

$$\int_{S^2} d^2 z |z|^{2k_1 \cdot k_2} |1-z|^{2k_2 \cdot k_3} = \sin(\pi k_2 \cdot k_3) \int_0^1 d\xi |\xi|^{k_1 \cdot k_2} |1-\xi|^{k_2 \cdot k_3} \int_1^\infty d\eta |\eta|^{k_1 \cdot k_2} |1-\eta|^{k_2 \cdot k_3}$$

  - Converting the second integral back to $\mathcal{A}_{\rm open}$, we obtain the KLT relation

$$\mathcal{A}^{(0)}_{\rm closed}(s,t,u) = \sin(\pi k_2 \cdot k_3) \mathcal{A}^{(0)}_{\rm open}(s,t) \mathcal{A}^{(0)}_{\rm open}(t,u)$$

  - Does the worldsheet secretly have a Minkowski signature structure ?
  - No generalization known to loop level